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A new technique to derive tight convex underestimators (sometimes envelopes)

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Abstract

The convex envelope value for a given function f over a region X at some point $\mathbf{x} \in X$ can be derived by searching for the largest value at that point among affine underestimators of f over X. This can be computed by solving a maximin problem, whose exact computation, however, may be a hard task. In this paper we show that by relaxation of the inner minimization problem, duality, and, in particular, by an enlargement of the class of underestimators (thus, not only affine ones) an easier derivation of good convex understimating functions, which can also be proved to be convex envelopes in some cases, is possible. The proposed approach is mainly applied to the derivation of convex underestimators (in fact, in some cases, convex envelopes) in the quadratic case. However, some results are also presented for polynomial, ratio of polynomials, and some other separable functions over regions defined by similarly defined separable functions.

Keywords Convex underestimators · Convex envelopes · Quadratic programming · Polynomial programming · Separable functions

1 Introduction

When dealing with nonconvex optimization problems, the derivation of convex underestimators for nonconvex functions over suitable regions enclosing the feasible one is an important issue. Indeed, nonconvex problems are often tackled through branch-and-bound approaches, whose performance heavily depends on the quality of the bounds. These are computed by solving convex relaxations, where convex underestimators replace nonconvex terms. Well known convex underestimators in the literature are

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those employed in the α-BB approaches: For a given function f and a box X, a convex underestimator is obtained by summing up f with a convex quadratic function q, with non positive values over X and whose Hessian has suitably large eigenvalues, making $f+q$ a convex function over X. The α-BB approaches have been introduced in [1] but many developments have been proposed later on, like the spline α-BB in [2] or a variant where function q is based on exponential functions, rather than quadratic ones, proposed in [3]. We also mention a variant where boxes are replaced by simplices in [4]. Another popular procedure to derive convex understimators is the Auxiliary Variable Method for factorable functions, i.e., functions obtained by a finite recursive combination of binary sums, binary products, and a library of univariate functions (like, e.g., logarithmic, exponential functions). An auxiliary variable replaces each factor of the factorable function, and an equality constraint establishes the equivalence between the auxiliary variable and the factor. Next, convex and concave envelopes of the factor are employed to approximate the equality constraint through two inequality constraints. We refer, e.g., to [5, 6].

Of course, the tighter the convex underestimator employed within a convex relaxation, the best is the lower bound returned by the relaxation. Thus, a special role is played by the *best* possible convex underestimator for a function f over a region X. This is called convex envelope and is defined as the pointwise supremum of all convex underestimators of f over X, i.e., after denoting by $\mathcal{C}(X)$ the set of convex functions over X, for any $\mathbf{x} \in X$ it holds that

$$Conv_{f,X}(\mathbf{x}) = \sup\{c(\mathbf{x}) \;\; c(\mathbf{y}) \leq f(\mathbf{y}) \; \forall \mathbf{y} \in X, \;\; c \in \mathcal{C}(X)\}. \tag{1}$$

In fact, it is well known that the above definition can be simplified by restricting the attention to affine underestimators. Indeed, each convex underestimating function can be replaced by an affine function (its supporting hyperplane at $\mathbf{x}$), and the two functions have the same value at $\mathbf{x}$. Thus, if we denote by $\mathcal{A}(X)$ the set of affine functions over X,

$$Conv_{f,X}(\mathbf{x}) = \sup\{c(\mathbf{x}) \;\; c(\mathbf{y}) \leq f(\mathbf{y}) \; \forall \mathbf{y} \in X, \;\; c \in \mathcal{A}(X)\}. \tag{2}$$

Computation of the convex envelope is a hard task. Indeed, since the global optimum value of $Conv_{f,X}$ over X is the same as the global optimum value of f over X, deriving the convex envelope is at least as hard as the global optimization of f over X. However, there are some results in the literature for some functions and regions. When X is a polytope, vertex polyhedral convex envelopes are those for which we can restrict the attention to piecewise linear functions, where each linear piece interpolates f at some vertices of the polytope (see, e.g., [7–10]). Non polyhedral convex envelopes are derived in different works including [11–18].

The equivalence between the two definitions (1) and (2) of the convex envelope leaves some freedom, which can be exploited. Indeed, for any set $\mathcal{H}(X)$ such that $\mathcal{A}(X) \subseteq \mathcal{H}(X) \subseteq \mathcal{C}(X)$, it holds that

$$Conv_{f,X}(\mathbf{x}) = \sup\{c(\mathbf{x}) \;\; c(\mathbf{y}) \leq f(\mathbf{y}) \; \forall \mathbf{y} \in X, \;\; c \in \mathcal{H}(X)\}. \tag{3}$$

Indeed, if we denote by $U_{\mathcal{C}}, U_{\mathcal{A}}, U_{\mathcal{H}}$ the supremum values in (1), (2), and (3), respectively, it holds that $U_{\mathcal{A}} \leq U_{\mathcal{H}} \leq U_{\mathcal{C}}$, and, in view of $U_{\mathcal{C}} = U_{\mathcal{A}}$, all the inequalities are, in fact, equalities. In particular, we define $\mathcal{H}(X)$ as follows

$$\mathcal{H}(X) = \{\boldsymbol{\alpha}^\top(\mathbf{y}-\mathbf{x}) + \eta + \tilde{f}(\mathbf{y}) - \tilde{f}(\mathbf{x}),\ \boldsymbol{\alpha} \in \mathbb{R}^n,\ \eta \in \mathbb{R},\ \tilde{f} \in \Phi_X\},$$

where $\Phi_X \subseteq \mathcal{C}(X)$. Thus, $\mathcal{H}(X) = \mathcal{A}(X)$ when $\Phi_X = \{0\}$, where 0 denotes the null function. Indeed, if $\tilde{f}$ reduces to the null function, then $\mathcal{H}(X)$ reduces to the set of affine functions. Instead, $\mathcal{H}(X) = \mathcal{C}(X)$ when $\Phi_X = \mathcal{C}(X)$. Indeed, $\mathcal{H}(X) \subseteq \mathcal{C}(X)$, since for each $\boldsymbol{\alpha} \in \mathbb{R}^n,\ \eta \in \mathbb{R},\ \tilde{f} \in \Phi_X$, it holds that $\boldsymbol{\alpha}^\top(\mathbf{y}-\mathbf{x}) + \eta + \tilde{f}(\mathbf{y}) - \tilde{f}(\mathbf{x})$ is a convex function. But at the same time it also holds that $\mathcal{C}(X) \subseteq \mathcal{H}(X)$. Indeed, each $f \in \mathcal{C}(X)$ can also be written in the form $\boldsymbol{\alpha}^\top(\mathbf{y}-\mathbf{x}) + \eta + \tilde{f}(\mathbf{y}) - \tilde{f}(\mathbf{x})$ by taking $\boldsymbol{\alpha} = \mathbf{0}$, $\eta = f(\mathbf{x})$, and $\tilde{f} = f$. Then, we can define the convex envelope as follows

$$\begin{aligned} Conv_{f,X}(\mathbf{x}) = \max_{\tilde{f}\in\Phi_X,\ \boldsymbol{\alpha},\ \eta}\ & \eta \\ & f(\mathbf{y}) \geq \boldsymbol{\alpha}^\top(\mathbf{y}-\mathbf{x}) + \eta + \tilde{f}(\mathbf{y}) - \tilde{f}(\mathbf{x})\ \forall \mathbf{y} \in X, \end{aligned}$$

or, equivalently

$$Conv_{f,X}(\mathbf{x}) = \max_{\tilde{f}\in\Phi_X,\ \boldsymbol{\alpha}} \boldsymbol{\alpha}^\top\mathbf{x} + \tilde{f}(\mathbf{x}) + \min_{\mathbf{y}\in X} f(\mathbf{y}) - \boldsymbol{\alpha}^\top\mathbf{y} - \tilde{f}(\mathbf{y}). \tag{4}$$

Now, given the inner minimization problem

$$\min_{\mathbf{y}\in X} f(\mathbf{y}) - \boldsymbol{\alpha}^\top\mathbf{y} - \tilde{f}(\mathbf{y}),$$

let

$$\max_{\mathbf{z}\in Z} \varphi(\mathbf{z}; f, \tilde{f}, \boldsymbol{\alpha}),$$

be a dual problem for it, where $\mathbf{z}$ are the dual variables and Z the feasible region of the dual problem. Note that an explicit derivation of the dual problem depends on the definition of f and $\tilde{f}$. The derivation will be given in the following sections for some special cases. Then, the optimal value of the following maximization problem

$$\max_{\tilde{f}\in\Phi_X,\ \boldsymbol{\alpha},\ \mathbf{z}\in Z} \boldsymbol{\alpha}^\top\mathbf{x} + \tilde{f}(\mathbf{x}) + \varphi(\mathbf{z}; f, \tilde{f}, \boldsymbol{\alpha}),$$

underestimates f at $\mathbf{x} \in X$. When the duality gap is null, this is the value of the convex envelope. The inner minimization problem can also be replaced by a relaxation. In this case the value of the convex envelope is obtained if the relaxation is exact and the duality gap is null. Obviously, due to the fact that computing the convex envelope of a function f over a given region X is at least as difficult as computing the global minimum of f over X, an efficient computation of the convex envelope is possible only when the global minimization problem can be efficiently solved, in particular for those nonconvex problems with a hidden convexity property. More precisely, it is possible only when the minimization over X of the sum of f with *any* linear function can be done efficiently (see the minimization inner problem in (4) with $\tilde{f} = 0$). Note that a polynomially solvable minimization problem may become hard to solve after the addition of a linear term. E.g., while the problem of minimizing the ratio of two linear functions over a polytope is polynomially solvable, the addition of a linear term makes it NP-hard (see [19]).

In the approach outlined above one can consider a minimal set of convex under-estimators, namely the affine ones or, equivalently, set $\Phi_X = \{0\}$. However, the main contribution of this paper is the following. Affine underestimators are enough in the definition of the convex envelope. However, the derivation of the convex envelope or, at least, of a good convex understimator, can be simplified by:

- exetnding the class of affine underestimators to include more general convex underestimators;
- exploiting duality.

The paper is structured as follows. In Sect. 2 we will extensively discuss how to apply the above observations in the quadratic case. In particular, we will:

- re-derive in a different way the convex underestimators of quadratic programming problems recently proposed in [20];
- discuss different cases when these convex underestimators are in fact convex envelopes, namely:
  - the case of diagonal Hessian matrices and no linear terms;
  - the case of a single strictly convex quadratic constraint together with some linear constraints, under a given rank condition;
  - some special structured problems with two quadratic constraints, related to unstructured robust least square problems.

Later on, in Sect. 3 we will discuss some convex underestimators for polynomial and ratio of polynomials functions, while in Sect. 4 we will discuss a generalization of the results about quadratic problems to some separable non quadratic problems.

2 Quadratic problems

In this section we discuss some cases involving quadratic functions. In these cases a rather natural choice for Φ_X is the set of all convex quadratic forms, i.e.,

$$\Phi_X = \{\mathbf{x}^\top \boldsymbol{\Gamma} \mathbf{x} \;\; \boldsymbol{\Gamma} \succeq \mathbf{O}\}. \tag{5}$$

Note that with this choice for Φ_X, $\mathcal{H}(X)$ turns out to be the set of convex quadratic functions. We will first show that by this choice we are able to derive in a different way the convex underestimators for QP problems proposed in [20]. The new derivation will also allow to identify cases when the derived convex underestimators are in fact the convex envelopes. Such cases will be discussed later on.

2.1 Convex underestimators for quadratic functions over quadratic constraints

We consider a function f and a region X defined as follows

$$\begin{aligned} & f(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}^\top \mathbf{Q}_0 \mathbf{x} \\ & X = \{\mathbf{x} \in \mathbb{R}^n \ \ \tfrac{1}{2}\mathbf{x}^\top \mathbf{Q}_i \mathbf{x} + \mathbf{p}_i^\top \mathbf{x} \le d_i, \ i = 1, \ldots, K\}. \end{aligned} \tag{6}$$

We make the following assumption.

Assumption 2.1 We assume that:

- at least one matrix $\mathbf{Q}_i$, $i = 1, \ldots, K$, is positive definite, so that X is a bounded set;
- $int(X) \neq \emptyset$, i.e., the interior of X is not empty.

Remark 2.1 In fact, the assumption that at least one matrix $\mathbf{Q}_i, i = 1, \ldots, K$, is positive definite, could be relaxed. It could be replaced by the more general requirement that function f is bounded from below over region X, which obviuosly holds true under Assumption 2.1 in view of Weierstrass' Theorem.

We prove the following result.

Theorem 2.1 *Under Assumption 2.1, a convex underestimator for f over X is*

$$\begin{aligned} U_{f,X}(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}^\top \mathbf{Q}_0 \mathbf{x} + \max_{\boldsymbol{\nu} \ge 0} \ & \textstyle\sum_{i=1}^K \nu_i \left[\tfrac{1}{2}\mathbf{x}^\top \mathbf{Q}_i \mathbf{x} + \mathbf{p}_i^\top \mathbf{x} - d_i\right] \\ & \mathbf{Q}_0 + \textstyle\sum_{i=1}^K \nu_i \mathbf{Q}_i \succeq \mathbf{O}. \end{aligned} \tag{7}$$

Proof By the previously proposed choice (5) for Φ_X, we have that (4) is

$$Conv_{f,X}(\mathbf{x}) = \max_{\boldsymbol{\Gamma} \succeq \mathbf{O}, \ \boldsymbol{\alpha}} \boldsymbol{\alpha}^\top \mathbf{x} + \mathbf{x}^\top \boldsymbol{\Gamma} \mathbf{x} + \min_{\mathbf{y} \in X} \tfrac{1}{2}\mathbf{y}^\top (\mathbf{Q}_0 - \boldsymbol{\Gamma})\mathbf{y} - \boldsymbol{\alpha}^\top \mathbf{y}. \tag{8}$$

We introduce the following $(n+1)$-dimensional square matrices

$$\mathbf{M}_0(\boldsymbol{\alpha}) = \frac{1}{2}\begin{pmatrix} \mathbf{Q}_0 - \boldsymbol{\Gamma} & -\boldsymbol{\alpha} \\ -\boldsymbol{\alpha}^\top & 0 \end{pmatrix}, \quad \mathbf{M}_i = \frac{1}{2}\begin{pmatrix} \mathbf{Q}_i & \mathbf{p}_i \\ \mathbf{p}_i^\top & 0 \end{pmatrix} \qquad i = 1, \ldots, K,$$

and

$$\mathbf{E}_{n+1,n+1} = \begin{pmatrix} \mathbf{O} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{pmatrix},$$

i.e., $\mathbf{E}_{n+1,n+1}$ is the matrix with a single unit component in the $(n+1, n+1)$ position, while all other components are null. Then, the following is a semidefinite relaxation for the inner minimization problem in (8)

$$\begin{array}{lll} \min_{\mathbf{Z}\succeq\mathbf{O}} & \mathbf{M}_0(\boldsymbol{\alpha})\bullet\mathbf{Z} & \\ & \mathbf{M}_i\bullet\mathbf{Z}\le d_i & i=1,\ldots,K \\ & \mathbf{E}_{n+1,n+1}\bullet\mathbf{Z}=1, & \end{array} \tag{9}$$

where

$$\mathbf{Z}=\begin{pmatrix}\mathbf{X} & \mathbf{x}\\ \mathbf{x}^\top & 1\end{pmatrix}.$$

Its dual is

$$\begin{array}{ll} \max_{\boldsymbol{\nu}\ge\mathbf{0},\ \beta} & -\sum_{i=1}^K \nu_i d_i+\beta \\ & \mathbf{M}_0(\boldsymbol{\alpha})+\sum_{i=1}^K \nu_i\mathbf{M}_i-\beta\mathbf{E}_{n+1,n+1}\succeq\mathbf{O}, \end{array} \tag{10}$$

so that the value $Conv_{f,X}(\mathbf{x})$ is underestimated by

$$\max_{\boldsymbol{\Gamma}\succeq\mathbf{O},\ \boldsymbol{\alpha}} \boldsymbol{\alpha}^\top\mathbf{x}+\mathbf{x}^\top\boldsymbol{\Gamma}\mathbf{x}+\begin{array}{ll}\max_{\boldsymbol{\nu}\ge\mathbf{0},\ \beta} & -\sum_{i=1}^K \nu_i d_i+\beta \\ & \mathbf{M}_0(\boldsymbol{\alpha})+\sum_{i=1}^K \nu_i\mathbf{M}_i-\beta\mathbf{E}_{n+1,n+1}\succeq\mathbf{O}.\end{array} \tag{11}$$

Recalling the definitions of $\mathbf{M}_0(\boldsymbol{\alpha})$, $\mathbf{M}_i$, $i=1,\ldots,K$, and $\mathbf{E}_{n+1,n+1}$, the inner maximization problem can be rewritten as follows

$$\begin{array}{ll} \max_{\boldsymbol{\nu}\ge\mathbf{0},\beta} & -\boldsymbol{\nu}^\top\mathbf{d}+\beta \\ & \begin{pmatrix}\mathbf{Q}_0+\sum_{i=1}^K \nu_i\mathbf{Q}_i-\boldsymbol{\Gamma} & -\boldsymbol{\alpha}+\sum_{i=1}^K \nu_i\mathbf{p}_i\\ (-\boldsymbol{\alpha}+\sum_{i=1}^K \nu_i\mathbf{p}_i)^\top & -2\beta\end{pmatrix}\succeq\begin{pmatrix}\mathbf{O} & \mathbf{0}\\ \mathbf{0}^\top & 0\end{pmatrix}. \end{array}$$

Using the Schur complement, we have that the matrix on the left-hand side of the semidefinite constraint is semidefinite positive if and only if either $\beta=0$, $-\boldsymbol{\alpha}+\sum_{i=1}^K \nu_i\mathbf{p}_i=\mathbf{0}$, and $\mathbf{Q}_0+\sum_{i=1}^K \nu_i\mathbf{Q}_i-\boldsymbol{\Gamma}\succeq\mathbf{O}$ or $\beta<0$ and

$$\mathbf{Q}_0+\sum_{i=1}^K \nu_i\mathbf{Q}_i-\boldsymbol{\Gamma}+\frac{(\boldsymbol{\alpha}-\sum_{i=1}^K \nu_i\mathbf{p}_i)(\boldsymbol{\alpha}-\sum_{i=1}^K \nu_i\mathbf{p}_i)^\top}{2\beta}\succeq\mathbf{O},$$

so that the problem is equivalent to

$$\begin{array}{ll} \max_{\boldsymbol{\nu}\ge\mathbf{0},\ \beta\le 0} & -\boldsymbol{\nu}^\top\mathbf{d}+\beta \\ & \mathbf{Q}_0+\sum_{i=1}^K \nu_i\mathbf{Q}_i+\frac{(\boldsymbol{\alpha}-\sum_{i=1}^K \nu_i\mathbf{p}_i)(\boldsymbol{\alpha}-\sum_{i=1}^K \nu_i\mathbf{p}_i)^\top}{2\beta}\succeq\boldsymbol{\Gamma}, \end{array}$$

where we set

$$\frac{(\boldsymbol{\alpha} - \sum_{i=1}^K \nu_i \mathbf{p}_i)(\boldsymbol{\alpha} - \sum_{i=1}^K \nu_i \mathbf{p}_i)^\top}{2\beta} = \mathbf{O},$$

when $\beta = 0$. Thus, by replacing $\boldsymbol{\Gamma}$ with

$$\mathbf{Q}_0 + \sum_{i=1}^K \nu_i \mathbf{Q}_i + \frac{(\boldsymbol{\alpha} - \sum_{i=1}^K \nu_i \mathbf{p}_i)(\boldsymbol{\alpha} - \sum_{i=1}^K \nu_i \mathbf{p}_i)^\top}{2\beta}, \tag{12}$$

the optimal value of (11) is equal to

$$\begin{aligned} \max_{\boldsymbol{\nu}\geq \mathbf{0},\boldsymbol{\alpha},\beta\leq 0} \quad & \frac{1}{2}\left(\mathbf{Q}_0 + \textstyle\sum_{i=1}^K \nu_i \mathbf{Q}_i\right) \bullet \mathbf{x}\mathbf{x}^\top + \left[\frac{(\boldsymbol{\alpha}-\sum_{i=1}^K \mathbf{p}_i)(\boldsymbol{\alpha}-\sum_{i=1}^K \nu_i\mathbf{p}_i)^\top}{4\beta}\right] \bullet \mathbf{x}\mathbf{x}^\top \\ & + (\boldsymbol{\alpha} - \textstyle\sum_{i=1}^K \nu_i \mathbf{p}_i)^\top \mathbf{x} + \sum_{i=1}^K \nu_i \mathbf{p}_i^\top \mathbf{x} - \boldsymbol{\nu}^\top \mathbf{d} + \beta \\ & \mathbf{Q}_0 + \textstyle\sum_{i=1}^K \nu_i \mathbf{Q}_i \succeq \mathbf{O}, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \max_{\boldsymbol{\nu}\geq \mathbf{0},\boldsymbol{\alpha},\beta\leq 0} \quad & \tfrac{1}{2}\mathbf{x}^\top \mathbf{Q}_0 \mathbf{x} + \textstyle\sum_{i=1}^K \nu_i \left[\tfrac{1}{2}\mathbf{x}^\top \mathbf{Q}_i \mathbf{x} + \mathbf{p}_i^\top \mathbf{x} - d_i\right] \\ & + \tfrac{[(\boldsymbol{\alpha}-\sum_{i=1}^K \mathbf{p}_i)^\top \mathbf{x}]^2}{4\beta} + (\boldsymbol{\alpha} - \textstyle\sum_{i=1}^K \nu_i \mathbf{p}_i)^\top \mathbf{x} + \beta \\ & \mathbf{Q}_0 + \textstyle\sum_{i=1}^K \nu_i \mathbf{Q}_i \succeq \mathbf{O}, \end{aligned}$$

and, finally, as

$$\begin{aligned} \max_{\boldsymbol{\nu}\geq \mathbf{0},\boldsymbol{\alpha},\beta\leq 0} \quad & \tfrac{1}{2}\mathbf{x}^\top \mathbf{Q}_0 \mathbf{x} + \textstyle\sum_{i=1}^K \nu_i \left[\tfrac{1}{2}\mathbf{x}^\top \mathbf{Q}_i \mathbf{x} + \mathbf{p}_i^\top \mathbf{x} - d_i\right] + \tfrac{1}{4\beta}\left[(\boldsymbol{\alpha} - \textstyle\sum_{i=1}^K \nu_i \mathbf{p}_i)^\top \mathbf{x} + 2\beta\right]^2 \\ & \mathbf{Q}_0 + \textstyle\sum_{i=1}^K \nu_i \mathbf{Q}_i \succeq \mathbf{O}. \end{aligned}$$

Thus, the maximum value is attained, e.g., when:

$$\left(\boldsymbol{\alpha} - \textstyle\sum_{i=1}^K \nu_i \mathbf{p}_i\right)^\top \mathbf{x} = \mathbf{0}, \quad \beta = 0,$$

and is equal to (7), as we wanted to prove. □

At this point it is worthwhile to stress the fact that the enlargement of the set of convex underestimators with (5) plays a relevant role. Indeed, the above development could be repeated with the smaller set of affine underestimators, obtained by setting $\Phi_X = \{0\}$, or, equivalently, imposing the constraint $\Gamma = \mathbf{O}$. However, in that case the derivation of the value of the convex underestimator would have not been straightforward. In fact, we could even hope that by enlarging the set of underestimators, the resulting value is also sharper. More precisely, let us denote by $U_{f,X}^q$ the convex underestimator when Φ_X is defined as in (5), and by $U^{\ell}_{f,X}$ the convex underestimator obtained when $\Phi_X = \{0\}$. It obviously holds that $U^{q}_{f,X}(\mathbf{x}) \geq U^{\ell}_{f,X}(\mathbf{x})$ for any $\mathbf{x} \in X$. We might wonder whether equality always holds or strict inequality holds at least for some $\mathbf{x} \in X$. Unfortunately, the former holds. This will be proved in a more general case in Sect. 3.

We also remark that the above convex underestimator is exactly the one presented in [20] but derived in a different way. The advantage of the new derivation is that it allows to easily detect subcases for which the convex underestimator is the best possible one, i.e., the convex envelope. Indeed, that happens when the relaxation (9) is exact and strong duality holds between (9) and (10). Some of these cases are discussed in the following Sect. 2.2.

2.2 Some cases where the convex underestimator is the convex envelope

In what follows we present some cases where the convex underestimator defined in (7) is in fact the convex envelope. According to the previous discussion, these are cases where the relaxation (9) is exact and strong duality holds for problem (9) and its dual (10).

2.2.1 Diagonal Hessian, no linear term

Let us assume that

$$\begin{aligned} \mathbf{Q}_i &= \mathbf{D}_i \;\; i = 0, \ldots, K \\ \mathbf{p}_i &= \mathbf{0} \quad i = 1, \ldots, K, \end{aligned} \tag{13}$$

where each matrix $\mathbf{D}_i$, $i = 0, \ldots, K$, is diagonal. Moreover, we restrict Φ_X as follows

$$\Phi_X = \{\mathbf{x}^\top \boldsymbol{\Gamma} \mathbf{x} : \; \boldsymbol{\Gamma} \succeq \mathbf{O}, \quad \boldsymbol{\Gamma} \text{ diagonal}\}. \tag{14}$$

In this case, $\mathcal{H}(X)$ turns out to be equivalent to the set of convex separable quadratic functions. The following results hold.

Proposition 2.1 *If* (13) *holds and* Φ_X *is defined as in* (14)*, then the semidefinite relaxation* (9) *is exact and, under Assumption* 2.1*, strong duality holds, i.e., also* (10) *is exact.*

Proof Exactness of the semidefinite relaxation (9) follows from Theorem 3.1 in [21] after observing that we can always take $\boldsymbol{\alpha} \geq \mathbf{0}$, while strong duality follows from Slater's condition. □

Corollary 2.1 *Under Assumption 2.1, the convex envelope of f over X, where f and X are defined according to (13), is*

$$
\begin{aligned}
Conv_{f,X}(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}^\top \mathbf{D}_0\mathbf{x} + \max_{\boldsymbol{\nu}\le \mathbf{0}} & \textstyle\sum_{i=1}^K \nu_i \left(d_i - \tfrac{1}{2}\mathbf{x}^\top \mathbf{D}_i\mathbf{x}\right) \\
& \mathbf{D}_0 - \textstyle\sum_{i=1}^K \nu_i \mathbf{D}_i \succeq \mathbf{O}.
\end{aligned}
$$

Proof It immediately follows from Theorem 2.1 and the results stated in Proposition 2.1. □

Note that in this case the computation of the value of the convex envelope amounts to the solution of a linear program.

Although we only discussed the case of quadratic functions with diagonal Hessian, we make the following remark which shows that the results extend also to cases with non-diagonal Hessian.

Remark 2.2 Let $K = 2$. If f and X are defined by (possibly) non-diagonal matrices $\mathbf{Q}_i, i = 0, 1, 2$, we can recover diagonality when the three matrices are simultaneously diagonalizable, i.e., when there exists a nonsingular matrix $\mathbf{S}$ such that

$$\mathbf{S}^\top \mathbf{Q}_i \mathbf{S} = \mathbf{D}_i, \quad i = 0, 1, 2,$$

where the matrices $\mathbf{D}_i$ are diagonal ones (see also [22]). Diagonality is recovered after the linear change of variables $\mathbf{z} = \mathbf{S}^{-1}\mathbf{x}$. Note that this include the case of two-sided constraints for which $\mathbf{Q}_1 = -\mathbf{Q}_2$, when $\mathbf{Q}_0$ and $\mathbf{Q}_1$ are simultaneously diagonalizable. Also note that the case $K = 1$ with $\mathbf{Q}_1$ positive definite is equivalent to the classical trust region problem. In this case we can take $\mathbf{Q}_2 = \mathbf{O}$ (and $d_2 > 0$) so that the three matrices are certainly simultaneously diagonalizable.

Remark 2.3 In [23] the convex envelope for the case $K = 1$ and $\mathbf{Q}_1$ not necessarily positive definite is derived, provided that there exists an interval $[\nu_-, \nu, +]$ such that $\mathbf{Q}_0 + \nu\mathbf{Q}_1 \succeq \mathbf{O}$ for each $\nu \in [\nu_-, \nu_+]$. This guarantees that function f is bounded over X. The convex envelope is

$$\frac{1}{2}\mathbf{x}^\top \mathbf{Q}_0\mathbf{x} + \frac{1}{2}\max\left\{\nu_-\left(\mathbf{x}^\top \mathbf{Q}_1\mathbf{x} - d_1\right), \nu_+\left(\mathbf{x}^\top \mathbf{Q}_1\mathbf{x} - d_1\right)\right\}. \tag{15}$$

In case $\mathbf{Q}_1$ is not positive definite we can not directly apply the results discussed in this paper, since Assumption 2.1 is not fulfilled. However, as pointed out in Remark 2.1, such assumption can be relaxed and, in fact, after observing that exactness of the semidefinite relaxation (9) is proved in Theorem 3.1 in [21] without requiring that matrix $\mathbf{D}_0$ and $\mathbf{D}_1$, obtained from $\mathbf{Q}_0$ and $\mathbf{Q}_1$ after simultaneous diagonalization, are postive definite, we can conclude that the convex envelope (15) can be derived as a special case of the results proved in this paper.

2.2.2 One quadratic and some linear constraints

Let us assume that

$$\begin{aligned} &\mathbf{Q}_1 = \mathbf{I} \\ &\mathbf{Q}_i = \mathbf{O} \ i = 2, \ldots, K \\ &\mathbf{p}_1 = \mathbf{0}, \end{aligned} \tag{16}$$

i.e., X is defined by a quadratic constraint and some linear constraints. Moreover, let us assume that $\mathbf{Q}_0$ fulfills the following condition, where $\lambda_{\min}(\mathbf{Q}_0)$ is the minimum eigenvalue of the symmetric matrix $\mathbf{Q}_0$

$$rank[\mathbf{Q}_0 - \lambda_{\min}(\mathbf{Q}_0)\mathbf{I}, \ \ \mathbf{p}_2, \ldots, \mathbf{p}_K] \leq n - 1. \tag{17}$$

This condition has been introduced in [24] and generalizes the *dimension condition* introduced in [25]

$$dim(Ker(\mathbf{Q}_0 - \lambda_{\min}(\mathbf{Q}_0)\mathbf{I})) \geq dim(span\{\mathbf{p}_2, \ldots, \mathbf{p}_K\}) + 1,$$

where $Ker(\mathbf{A})$ denotes the kernel of matrix $\mathbf{A}$. We also restrict Φ_X as follows i.e.,

$$\Phi_X = \{\mathbf{x}^\top \mathbf{\Gamma} \mathbf{x} \ \ \mathbf{\Gamma} \succeq \mathbf{O}, \quad \text{and} \quad \mathbf{Q}_0 - \mathbf{\Gamma} \text{ fulfills condition } (17)\}. \tag{18}$$

We prove the following results.

Proposition 2.2 *If* (16) *holds and* Φ_X *is defined as in* (18)*, then the semidefinite relaxation* (9) *is exact and, under Assumption* 2.1*, strong duality holds, i.e., also* (10) *is exact.*

Proof Exactness of the semidefinite relaxation under condition (17) has been proved in [24]. Strong duality follows from Slater's condition. □

The following is a corollary of Theorem 2.1 and of the strong duality result stated in Proposition 2.2.

Corollary 2.2 *Under Assumption* 2.1*, the convex envelope of* f *over* X*, where* f *and* X *are defined according to* (16)*, is*

$$Conv_{f,X}(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top \mathbf{Q}_0 \mathbf{x} + \frac{1}{2}\lambda_{min}(\mathbf{Q}_0)\left(1 - \mathbf{x}^\top \mathbf{x}\right).$$

It is worthwhile to remark that the convex envelope is the same one as the convex envelope of f over the unit sphere. In other words, the addition of linear constraints for which condition (17) holds does not change the convex envelope.

2.2.3 Special structured problems with two quadratic constraints

Let us assume that

$$\mathbf{Q}_i = \mathbf{I}_r \otimes \mathbf{A}_i \;\; i = 0, 1, 2, \tag{19}$$

where $\mathbf{I}_r$ is the r-dimensional identity matrix, $r > 1$, $\mathbf{A}_i$, $i = 0, 1, 2$, are m-dimensional symmetric matrices, $\otimes$ denotes the Kronecker product, and $n = rm$. In [26] it is shown that applications with Hessian matrices defined as in (19) arise from unstructured robust least squares problems. We restrict Φ_X as follows

$$\Phi_X = \{\mathbf{x}^\top [\mathbf{I}_r \otimes \mathbf{\Gamma}]\, \mathbf{x} \;\; \mathbf{\Gamma} \succeq \mathbf{O}\}. \tag{20}$$

We prove the following results.

Proposition 2.3 *If (19) holds and Φ_X is defined as in (20), then the semidefinite relaxation (9) is exact and, under Assumption 2.1, strong duality holds, i.e., also (10) is exact.*

Proof Exactness of the semidefinite relaxation when (19) holds has been proved in [26]. Strong duality follows from Slater's condition. □

The following is again a corollary of Theorem 2.1 and of the strong duality result stated in Proposition 2.3.

Corollary 2.3 *Under Assumption 2.1. the convex envelope of f over X, where f and X are defined according to (19), is*

$$\begin{aligned} Conv_{f,X}(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}^\top [\mathbf{I}_r \otimes \mathbf{A}_0]\, \mathbf{x} + \max_{\nu_1,\nu_2 \le 0} \; & \nu_1 \left(d_1 - \tfrac{1}{2}\mathbf{x}^\top [\mathbf{I}_r \otimes \mathbf{A}_1]\, \mathbf{x} - \mathbf{p}_1^\top \mathbf{x}\right) \\ & + \nu_2 \left(d_2 - \tfrac{1}{2}\mathbf{x}^\top [\mathbf{I}_r \otimes \mathbf{A}_2]\, \mathbf{x} - \mathbf{p}_2^\top \mathbf{x}\right) \; . \\ & \mathbf{A}_0 - \nu_1 \mathbf{A}_1 - \nu_2 \mathbf{A}_2 \succeq \mathbf{O}. \end{aligned}$$

3 Polynomials and ratio of polynomials

Following [27], let us consider a function f which is the ratio $\frac{p}{q}$ of two polynomial functions, and let X be defined as follows

$$X = \{\mathbf{x} \in \mathbb{R}^n \;\; g_i(\mathbf{x}) \ge 0, \;\; i = 1, \dots, m\},$$

where each g_i is a polynomial function. We assume that X has a nonempty interior and that $q > 0$ over X. We add the constraint $g_{m+1}(\mathbf{x}) = q(\mathbf{x}) \ge 0$ without modifying X. After setting $\tilde{f} = 0$ (equivalent to setting $\Phi_X = \{0\}$), we can rewrite (4) as follows

$$Conv_{f,X}(\mathbf{x}) = \max \min_{\mathbf{z} \in X} f(\mathbf{z}) - \boldsymbol{\alpha}^\top (\mathbf{z} - \mathbf{x}),$$

which can be further rewritten as

$$
\begin{aligned}
Conv_{f,X}(\mathbf{x}) = \sup_{\gamma,\boldsymbol{\alpha}} \ & \gamma + \boldsymbol{\alpha}^\top \mathbf{x} \\
& p(\mathbf{z}) - \gamma q(\mathbf{z}) - \left(\boldsymbol{\alpha}^\top \mathbf{z}\right) q(\mathbf{z}) \geq 0 \ \ \forall\, \mathbf{z} \in X.
\end{aligned}
\tag{21}
$$

However, the constraint imposing nonnegativity over X is a difficult one. Thus, in [27] a sequence of convex underestimators for f is defined as follows for each $\mathbf{x} \in X$

$$
\begin{aligned}
f^r(\mathbf{x}) = \sup_{\gamma,\boldsymbol{\alpha},u_i} \ & \gamma + \boldsymbol{\alpha}^\top \mathbf{x} \\
& p(\mathbf{z}) - \gamma q(\mathbf{z}) - \left(\boldsymbol{\alpha}^\top \mathbf{z}\right) q(\mathbf{z}) = u_0(\mathbf{z}) + \textstyle\sum_{i=1}^{m+1} u_i(\mathbf{z}) g_i(\mathbf{z}) \ \ \forall \mathbf{z} \\
& u_0 \in \Sigma_n^{2r}, \ u_i \in \Sigma_n^{2r-deg(g_i)},
\end{aligned}
\tag{22}
$$

where deg denotes the degree of a polynomial function and Σ_n^d denotes the set of polynomials in n variables with degree at most d and which can be written as sum of squares of polynomials (also called in what follows *sos* polynomials). It is well known that *sos* polynomials can be represented by semidefinite matrices, so that the evaluation of f^r can be done through the solution of a semidefinite programming problem, whose size increases (rapidly) with r. We remark that if the constraint in (22) is fulfilled, then the constraint in (21) is also fulfilled (while the vice versa is not necessarily true). Thus, the value $f^r(\mathbf{x})$ underestimates $Conv_{f,X}(\mathbf{x})$. We remark that when p and g_i, $i = 1, \ldots, m$, are quadratic functions and $q(\mathbf{z}) = 1$ $\forall \mathbf{z}$, $f^1 \equiv U_{f,X}^{\ell}$, where $U_{f,X}^{\ell}$ has been defined in Sect. 2.1. In [27] it is proved that under mild assumptions on X (fulfilled, e.g., when X is a polytope or at least one region $\{\mathbf{x} \in \mathbb{R}^n \ \ g_i(\mathbf{x}) \geq 0\}$, $i = 1, \ldots, m$, is compact), the sequence $\{f^r(\mathbf{x})\}$ converges pointwise to $Conv_{f,X}(\mathbf{x})$ as $r \to \infty$ for each $\mathbf{x} \in X$.

We might wonder whether an enlargement of the set of convex underestimators allows for the detection of a sharper value with respect to f^r. We need to enlarge the set of convex underestimators in such a way that the new value can still be efficiently computed. We define Φ_X^r as follows

$$
\begin{aligned}
\Phi_X^r = \{s \ \ & s \text{ is sos-convex}, \ deg(s) \leq 2r - deg(q), \\
& \text{no monomials with degree lower than 2 in } s\}.
\end{aligned}
$$

A polynomial $s(\mathbf{x})$ is *sos-convex* if its Hessian $H(\mathbf{x})$ is an *sos-matrix*. Sos-convexity is a sufficient but not necessary condition for convexity of a polynomial. Its nice feature is that it can be checked through a semidefinite condition, while checking convexity is a hard problem even for polynomials of degree 4 (see [28]). Note that we imposed that monomials in s have degree not lower than two since affine terms are dealt with separately. Also note that when $q(\mathbf{z}) = 1$ $\forall \mathbf{z}$, Φ_X^1 is exactly the set defined in (5).

Thus, we could define a new underestimator as follows

$$h^r(\mathbf{x}) = \sup_{\gamma,\boldsymbol{\alpha},s,u_i} \gamma + \boldsymbol{\alpha}^\top \mathbf{x} + s(\mathbf{x})$$
$$p(\mathbf{z}) - \gamma q(\mathbf{z}) - (\boldsymbol{\alpha}^\top \mathbf{z})q(\mathbf{z}) - s(\mathbf{z})q(\mathbf{z}) = u_0(\mathbf{z}) + \sum_{i=1}^{m+1} u_i(\mathbf{z})g_i(\mathbf{z}) \ \ \forall \mathbf{z} \tag{23}$$
$$u_0 \in \Sigma_n^{2r}, \ u_i \in \Sigma_n^{2r-deg(g_i)}, \ s \in \Phi_X^r.$$

We obviously have that $h^r(\mathbf{x}) \geq f^r(\mathbf{x})$ for any $\mathbf{x} \in X$. Indeed, from any feasible solution of (23) with $s = 0$, we can always derive a feasible solution for (22) with the same objective function value. Unfortunately, the following proposition proves that equality always holds.

Proposition 3.1 *It holds that $h^r(\mathbf{x}) = f^r(\mathbf{x})$ for any $\mathbf{x} \in X$.*

Proof As already observed, $h^r(\mathbf{x}) \geq f^r(\mathbf{x})$. Then, if we are able to prove the reverse inequality, we are done. Let $\left(\boldsymbol{\alpha}^*, \gamma^*, u_0^*, u_1^*, \ldots, u_m^*, u_{m+1}^*, s^*\right)$ be an optimal solution of (23), whose optimal value is, thus, $h^r(\mathbf{x}) = \gamma^* + \boldsymbol{\alpha}^{*T}\mathbf{x} + s^*(\mathbf{x})$. By feasibility, it holds that $\forall \mathbf{z}$

$$p(\mathbf{z}) - \gamma^* q(\mathbf{z}) - (\boldsymbol{\alpha}^{*T}\mathbf{z})q(\mathbf{z}) = u_0^*(\mathbf{z}) + \sum_{i=1}^{m} u_i^*(\mathbf{z})g_i(\mathbf{z}) + (u_{m+1}^*(\mathbf{z}) + s^*(\mathbf{z}))g_{m+1}(\mathbf{z}),$$

which can be further rewritten as

$$f(\mathbf{z}) - \gamma^* q(\mathbf{z}) - (\boldsymbol{\alpha}^{*T}\mathbf{z})q(\mathbf{z}) - (s^*(\mathbf{x}) + \nabla s^*(\mathbf{x})^\top(\mathbf{z} - \mathbf{x}))q(\mathbf{z})$$
$$= u_0^*(\mathbf{z}) + \sum_{i=1}^{m} u_i^*(\mathbf{z})g_i(\mathbf{z}) + (u_{m+1}^*(\mathbf{z}) + s^*(\mathbf{z}) - s^*(\mathbf{x}) - \nabla s^*(\mathbf{x})^\top(\mathbf{z} - \mathbf{x}))g_{m+1}(\mathbf{z}),$$

after subtracting $(s^*(\mathbf{x}) + \nabla s^*(\mathbf{x})^\top(\mathbf{z} - \mathbf{x}))q(\mathbf{z})$ to both sides of the equation (recall that $g_{m+1} = q$). Now, let $g^*(\mathbf{z}) = s^*(\mathbf{z}) - s^*(\mathbf{x}) - \nabla s^*(\mathbf{x})^\top(\mathbf{z} - \mathbf{x})$. We notice that g^* is an sos-convex function. Indeed, since g^* and s^* only differ by a linear term, the Hessian matrix of g^* is the same as the one of s^*, which is an sos-matrix. Moreover, it holds that $g^*(\mathbf{x}) = 0$ and $\nabla g^*(\mathbf{x}) = \mathbf{0}$. Thus, according to Lemma 8 in [29], it turns out that $g^* \in \Sigma_n^{2r-deg(q)}$. Then,

$$\left(\boldsymbol{\alpha}^* + \nabla s^*(\mathbf{x}), \gamma^* + s^*(\mathbf{x}) - \nabla s^*(\mathbf{x})^\top\mathbf{x}, u_0^*, u_1^*, \ldots, u_m^*, u_{m+1}^* + g^*\right),$$

is a feasible solution for (22) with objective function value equal to $\gamma^* + \boldsymbol{\alpha}^{*T}\mathbf{x} + s^*(\mathbf{x})$. Consequently, $h^r(\mathbf{x}) \leq f^r(\mathbf{x})$, as we wanted to prove. □

As a corollary of this result, we also have the equivalence between $U_{f,X}^q$ and $U_{f,X}^\ell$ as defined at the end of Sect. 2.1. Indeed, $h^1 \equiv U_{f,X}^q$ in view of the equivalence between Φ_X^1 and (5), while, as already observed, $f^1 \equiv U_{f,X}^\ell$.

Functions f^r and h^r are computed via the solution of semidefinite programs and, consequently, we do not have their analytic expression. However, in [27] it is observed that the optimal solution of (22) allows to define an affine underestimator for f. More precisely, given an optimal solution $\left(\boldsymbol{\alpha}^*, \gamma^*, u_0^*, u_1^*, \ldots, u_m^*, u_{m+1}^*\right)$ of (22), it turns out that a supporting hyperplane for f^r at $\mathbf{x}$ and, thus, an affine underestimator for f over X, is

$$\ell_{\mathbf{x}}(\mathbf{z}) = \gamma^* + \boldsymbol{\alpha}^{*T}\mathbf{z}.$$

Similarly, given an optimal solution $\left(\boldsymbol{\alpha}^*, \gamma^*, u_0^*, u_1^*, \ldots, u_m^*, u_{m+1}^*, s^*\right)$ for (23), we have that

$$c_{\mathbf{x}}(\mathbf{z}) = \gamma^* + \boldsymbol{\alpha}^{*T}\mathbf{z} + s^*(\mathbf{z}),$$

is a convex underestimator for f over X. While it obviously holds that $f^r(\mathbf{x}) = \ell_{\mathbf{x}}(\mathbf{x}) = c_{\mathbf{x}}(\mathbf{x}) = h^r(\mathbf{x})$, for some other $\mathbf{z} \in X$, $c_{\mathbf{x}}(\mathbf{z}) > \ell_{\mathbf{x}}(\mathbf{z})$ may hold. Thus, the convex underestimator returned by the solution of (23) may be tighter than the affine one returned by the solution of (22). This is shown through the following example.

Example 3.1 Let us consider the following function taken from [27]

$$f(x_1, x_2) = \frac{x_1 x_2}{1 + x_1^2 + x_2^2},$$

with $X = [-1, 1]^2$ or, equivalently

$$X = \{(x_1, x_2) \;\; g_1(x_1, x_2) = 1 - x_1^2 \geq 0, \;\; g_2(x_1, x_2) = 1 - x_2^2 \geq 0\}.$$

It is worthwhile to remark that, although this is a two-dimensional function, the computation of its convex envelope is not trivial at all. To the author's knowledge, techniques for the derivation of the convex envelope can be applied when the function is edge-concave over a polyhedral region X (i.e., concave along any direction parallel to the edges of X), in which case the convex envelope is vertex polyhedral (see, e.g., [7, 9]), or when the Hessian of f is not positive semidefinite in the interior of X (see, e.g., [12, 14, 15]). None of these conditions is fulfilled by function f over the box X.

Let $\mathbf{x} = (0.1, -0.1)$. After solving (22) and (23) with $r = 2$ we get

$$\ell_{\mathbf{x}}(x_1, x_2) = -\frac{1}{3}, \quad c_{\mathbf{x}}(x_1, x_2) = -\frac{1}{3} + 0.0426 * (x_1 + x_2)^2.$$

The original function f, $\ell_{\mathbf{x}}$, and $c_{\mathbf{x}}$ are displayed in Fig. 1.

In fact, the convex underestimator can be in some cases considerably improved by the following observation. Problem (23) may have multiple optimal solutions and the corresponding functions s^* may considerably differ from each other far from the point $\mathbf{x}$. Thus, we could include in the objective function of (23) some term which favors optimal solutions with a larger value of s^* in points different from $\mathbf{x}$. Thus, we could

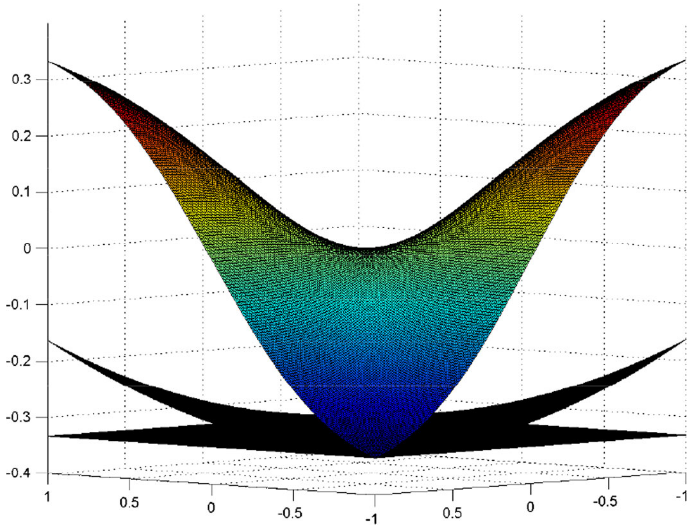

Fig. 1 Functions f, $\ell_{\mathbf{x}}$, and $c_{\mathbf{x}}$

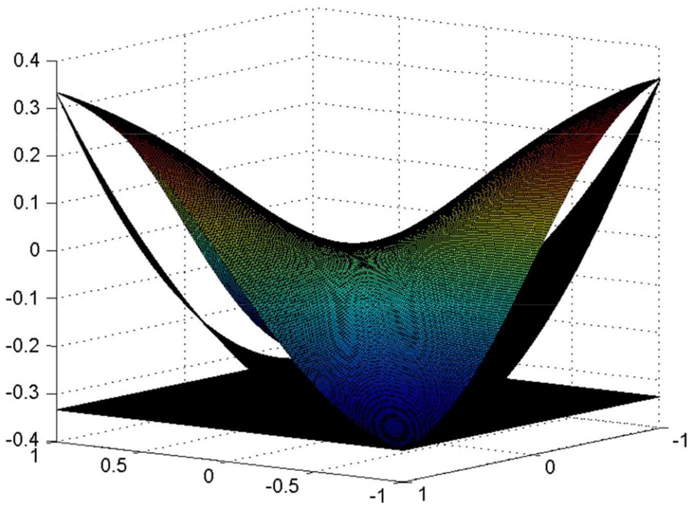

Fig. 2 Functions f, $\ell_{\mathbf{x}}$, and $c'_{\mathbf{x}}$

fix some points $\mathbf{v}_1, \ldots, \mathbf{v}_K \in X$ (e.g., the vertices of X if it is a box), fix a small value $\varepsilon > 0$, and add the term $\varepsilon \sum_{j=1}^{K} s(\mathbf{v}_j)$ to the objective function in (23). In our example, such simple modification leads to the convex underestimator

$$c'_{\mathbf{x}}(x_1, x_2) = -\frac{1}{3} + \frac{1}{6} * (x_1 + x_2)^2,$$

also displayed in Fig. 2, which is much tighter than the one displayed in Fig. 1.

Finally, we make the following observation. By solving (22) or (23) we compute a (possibly different) convex underestimator for each $\mathbf{x} \in X$. If we solve multiple instances of these problems for different points in X, we obtain a piecewise defined convex underestimator, piecewise linear in case of (22) and piecewise sos-convex in case of (23). However, if we wish to solve a single problem, then we can in some cases proceed as follows. The quality of an underestimator c for a function f over a region X can be measured by the volume of the region between the graphs of the two functions, i.e., by the integral $\int_X (f - c)$. Since f is given, this amounts to maximizing $\int_X c$ among all possible underestimators within a given class. In case X is a box and the class of underestimators is made up of polynomials, the integral is easy to compute. For a polynomial function f and a box X, Lasserre and Thanh [30] propose to maximize this quality measure over the class of polynomials with degree d which are convex over X, i.e., to solve the following problem

$$\begin{aligned} \max\ & \textstyle\int_X c(\mathbf{x}) d\mathbf{x} \\ & f(\mathbf{z}) \geq c(\mathbf{z}) \quad \forall \mathbf{z} \in X \\ & c \text{ is a degree-}d \text{ polynomial, convex over } X. \end{aligned} \tag{24}$$

Convexity over X of a polynomial c corresponds to nonnegativity of the polynomial $h(\mathbf{x}, \mathbf{y}) = \mathbf{y}^\top \nabla^2 c(\mathbf{x}) \mathbf{y}$ over $X \times U$, where U is the n-dimensional unit sphere. Nonnegativity both of $f - c$ and of h are hard constraints, so that computing an optimal solution $\bar{f}_d$ of (24) is a difficult task. Thus, in [30] a sequence of semidefinite programming problems of increasing size is proposed, and it is shown that the sequence of their optimal solutions $\{\bar{f}_{dk}\}$ is a sequence of convex underestimators for f with degree d, and that each accumulation point of the sequence is an optimal solution of (24). Checking sos-convexity of a polynomial is much simpler than checking its convexity over X. Indeed, it is enough to impose that $h(\mathbf{x}, \mathbf{y})$ is an sos polynomial, which reduces to a semidefinite condition. Thus, a simpler alternative to (24), requiring the solution of a single semidefinite programming problem, is that of replacing nonnegativity of $f - c$ with an sos condition, and convexity of c over X with an sos-convexity condition. That leads to the following problem

$$\begin{aligned} \sup_{\gamma, \boldsymbol{\alpha}, s, u_i}\ & \textstyle\int_X (\gamma + \boldsymbol{\alpha}^\top \mathbf{x} + s(\mathbf{x})) d\mathbf{x} \\ & f(\mathbf{z}) - \gamma q(\mathbf{z}) - (\boldsymbol{\alpha}^\top \mathbf{z}) q(\mathbf{z}) - s(\mathbf{z}) q(\mathbf{z}) = u_0(\mathbf{z}) + \textstyle\sum_{i=1}^{m+1} u_i(\mathbf{z}) g_i(\mathbf{z}) \ \ \forall \mathbf{z} \\ & u_0 \in \Sigma_n^d,\ u_i \in \Sigma_n^{d-deg(g_i)},\ s \in \Phi_X^{\frac{d}{2}}. \end{aligned} \tag{25}$$

In our example, when $d = 4$ it turns out that the solution of the problem is exactly function $c'_{\mathbf{x}}$.

If we solve (23) we search for a convex underestimator which is as tight as possible at a single point $\mathbf{x} \in X$. If we solve (25) we search for a convex underestimator which is as tight as possible over the whole box X. It is possible to define intermediate underestimators by changing the objective function in (25) as follows

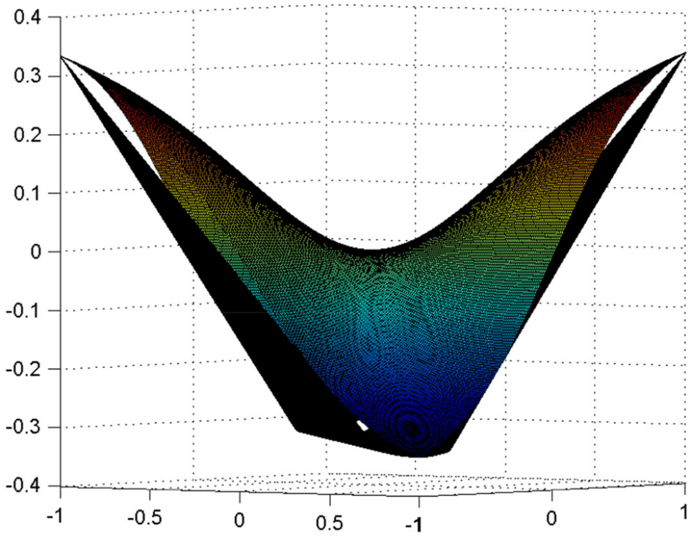

Fig. 3 Function f and the convex underestimator $\max\{c_{X_1}, c_{X_2}, c_{X_4}\}$

$$\int_{X'} (\gamma + \boldsymbol{\alpha}^\top \mathbf{x} + s(\mathbf{x}))d\mathbf{x}, \tag{26}$$

for some $X' \subseteq X$ (in fact, (23) can be seen as a special case where $X' = \{\mathbf{x}\}$ and (25) as a special case where $X' = X$). Thus, we could introduce a piecewise defined convex underestimator (with a prefixed number of pieces) as follows: (i) select a collection $\{X_i\}_{i=1}^K$ of subsets of X which are either boxes or union of boxes so that the integral over them is a linear function of the polynomial coefficients; (ii) compute a different convex underestimator for f over X for each X_i in the collection by solving (25) with objective function (26) and with $X' = X_i$; (iii) take the maximum of these convex underestimators. For instance, in our example we consider a collection of subsets of $[-1, 1]^2$ made up of four corner squares and a central square:

$$X_1 = [-0.5, 0.5]^2., \quad X_2 = [0.5, 1]^2, \quad X_3 = [0.5, 1] \times [-1, -0.5],$$
$$X_4 = [-1, -0.5]^2, \quad X_5 = [-1, -0.5] \times [0.5, 1].$$

The resulting convex underestimators are

$$c_{X_1}(x_1, x_2) = c_{X_3}(x_1, x_2) = c_{X_5}(x_1, x_2) = -\tfrac{1}{3} + \tfrac{1}{6} * (x_1 + x_2)^2$$
$$c_{X_2}(x_1, x_2) = -0.4082 + 0.3708 * (x_1 + x_2)$$
$$c_{X_4}(x_1, x_2) = -0.4082 - 0.3708 * (x_1 + x_2),$$

and the overall convex underestimator is displayed in Fig. 3.

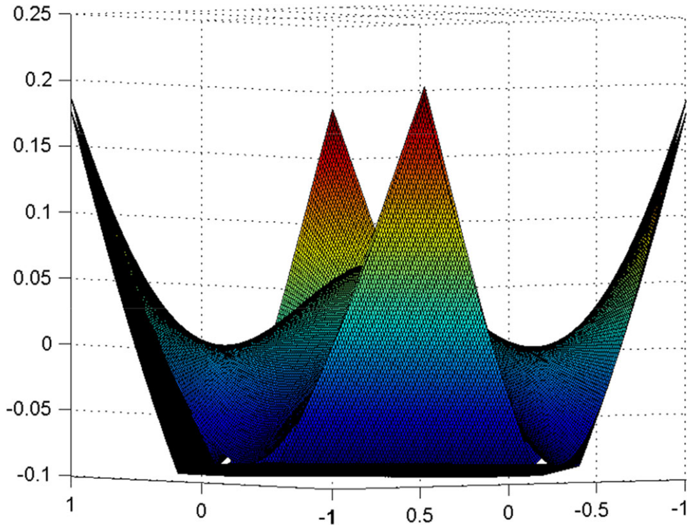

Fig. 4 Function g and the convex underestimator $\max\{c_{X_1}, c_{X_2}, c_{X_3}, c_{X_4}, c_{X_5}\}$

As a further example, we might consider the following function also taken from [27]

$$g(x_1, x_2) = \frac{\left(x_1^2 - \frac{1}{4}\right)\left(x_2^2 - \frac{1}{4}\right)}{1 + x_1^2 + x_2^2},$$

again over the box $X = [-1, 1]^2$. In this case the solution of (25) is the trivial constant underestimator $c_X(x_1, x_2) = -0.0937$, corresponding to the minimum of g over X. If we consider the same collection of subests as before we end up with the following convex underestimators

$$c_{X_1}(x_1, x_2) = -0.0937$$
$$c_{X_2}(x_1, x_2) = -0.2897 + 0.0686 * (x_1 + x_2) + 0.0826 * (x_1 + x_2)^2$$
$$c_{X_3}(x_1, x_2) = -0.2897 + 0.0686 * (x_1 - x_2) + 0.0826 * (x_1 - x_2)^2$$
$$c_{X_4}(x_1, x_2) = -0.2897 - 0.0686 * (x_1 + x_2) + 0.0826 * (x_1 + x_2)^2$$
$$c_{X_5}(x_1, x_2) = -0.2897 + 0.0686 * (x_2 - x_1) + 0.0826 * (x_2 - x_1)^2.$$

Function g and the resulting convex underestimator are displayed in Fig. 4.

4 General separable problems

The approach discussed in Sect. 2.1 for quadratic problems can be extended to some special structured non quadratic problems. Let us consider a n-dimensional vector of one-dimensional functions

$$\mathbf{F}(\mathbf{x}) = (f_1(x_1), \ldots, f_n(x_n)),$$

where each function f_i is strictly convex, differentiable and coercive, i.e.,

$$\lim_{x_i \to \pm\infty} f_i(x_i) = +\infty.$$

We denote by

$$\mathbf{F}'(\mathbf{x}) = (f_1'(x_1), \ldots, f_n'(x_n)),$$

the n-dimensional vector of derivatives of the one-dimensional functions f_i, $i = 1, \ldots, n$. We denote by $\mathbf{x}^*$ the vector whose ith component x_i^* is the minimum of function f_i. Moreover, let us consider a k-dimensional vector of n-dimensional functions

$$\mathbf{G}(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_k(\mathbf{x})),$$

where each function g_j, $j = 1, \ldots, k$, is convex and differentiable. Let

$$f(\mathbf{x}) = h(\mathbf{F}(\mathbf{x})),$$

where h is a convex differentiable function, and

$$X = \{\mathbf{x} \in \mathbb{R}^n \;\; \mathbf{G}(\mathbf{F}(\mathbf{x})) \leq \mathbf{m}\}, \quad \mathbf{m} \in \mathbb{R}^k.$$

We assume that $int(X) \neq \emptyset$. Note that the quadratic case discussed in Sect. 2.2.1 is a special case of the one discussed in this section where:

- $f_i(x_i) = x_i^2, i = 1, \ldots, n$;
- $h(\mathbf{t}) = \boldsymbol{\mu}^\top \mathbf{t}$;
- $\mathbf{G}(\mathbf{t}) = \mathbf{M}\mathbf{t}$, where the ith row of matrix $\mathbf{M}$ is $diag(\mathbf{D}_i)$, i.e., the vector whose components are the diagonal elements of $\mathbf{D}_i$.

In this case we define the set Φ_X of convex functions as follows

$$\Phi_X = \{\tilde{f} : \tilde{f}(\mathbf{x}) = \boldsymbol{\delta}^\top \mathbf{F}(\mathbf{x}), \quad \boldsymbol{\delta} \geq \mathbf{0}\}. \tag{27}$$

In this case, $\mathcal{H}(X)$ is the set of functions obtained by summing up affine functions with nonnegative combinations of the convex functions f_i, $i = 1, \ldots, n$. The following result holds.

Proposition 4.1 *For any $\boldsymbol{\delta} \geq \mathbf{0}$, the minimization problem*

$$\begin{aligned} \min\ & h(\mathbf{F}(\mathbf{y})) - \boldsymbol{\delta}^\top \mathbf{F}(\mathbf{y}) - \boldsymbol{\alpha}^\top (\mathbf{y} - \mathbf{x}^*) \\ & \mathbf{G}(\mathbf{F}(\mathbf{y})) \leq \mathbf{m}, \end{aligned}$$

admits the strong dual

$$\begin{aligned} \max\ & h(\mathbf{t}) - (\boldsymbol{\delta}+\boldsymbol{\nu})^\top \mathbf{t} + \boldsymbol{\gamma}^\top \mathbf{G}(\mathbf{t}) - \boldsymbol{\gamma}^\top \mathbf{m} + \boldsymbol{\nu}^\top \mathbf{F}\left(\mathbf{F}'^{-1}\left(\tfrac{\boldsymbol{\alpha}}{\boldsymbol{\nu}}\right)\right) - \boldsymbol{\alpha}^\top \mathbf{F}'^{-1}\left(\tfrac{\boldsymbol{\alpha}}{\boldsymbol{\nu}}\right) + \boldsymbol{\alpha}^\top \mathbf{x}^* \\ & \boldsymbol{\delta} = \boldsymbol{\gamma}^\top \nabla \mathbf{G}(\mathbf{t}) + \nabla h(\mathbf{t}) + \boldsymbol{\nu} \\ & \boldsymbol{\nu}, \boldsymbol{\gamma} \geq \mathbf{0}, \end{aligned} \tag{28}$$

where $\nabla \mathbf{G}(\mathbf{t})$ *is the* $k \times n$ *dimensional matrix whose* i*th row is* $\nabla g_i(\mathbf{t})$.

Proof The problem admits the following convex exact reformulation

$$\begin{aligned} \min\ & h(\mathbf{t}) - \boldsymbol{\delta}^\top \mathbf{t} - \boldsymbol{\alpha}^\top (\mathbf{y} - \mathbf{x}^*) \\ & \mathbf{G}(\mathbf{t}) \leq \mathbf{m} \\ & \mathbf{F}(\mathbf{y}) \leq \mathbf{t}. \end{aligned}$$

Exactness of this relaxation has been proved in [31]. Since $int(X) \neq \emptyset$, Slater's condition holds and, consequently, the Lagrangian dual

$$\max_{\boldsymbol{\nu},\boldsymbol{\gamma}\geq\mathbf{0}} \min_{\mathbf{y},\mathbf{t}} h(\mathbf{t}) - (\boldsymbol{\delta}+\boldsymbol{\nu})^\top \mathbf{t} + \boldsymbol{\gamma}^\top \mathbf{G}(\mathbf{t}) - \boldsymbol{\gamma}^\top \mathbf{m} - \boldsymbol{\alpha}^\top (\mathbf{y} - \mathbf{x}^*) + \boldsymbol{\nu}^\top \mathbf{F}(\mathbf{y}),$$

is exact. The minimum with respect to $\mathbf{t}$ is attained when

$$\boldsymbol{\gamma}^\top \nabla \mathbf{G}(\mathbf{t}) = -\nabla h(\mathbf{t}) + \boldsymbol{\nu} + \boldsymbol{\delta}.$$

while

$$\min_{\mathbf{y}} -\boldsymbol{\alpha}^\top \mathbf{y} + \boldsymbol{\nu}^\top \mathbf{F}(\mathbf{y}) = \boldsymbol{\nu}^\top \mathbf{F}\left(\mathbf{F}'^{-1}\left(\frac{\boldsymbol{\alpha}}{\boldsymbol{\nu}}\right)\right) - \boldsymbol{\alpha}^\top \mathbf{F}'^{-1}\left(\frac{\boldsymbol{\alpha}}{\boldsymbol{\nu}}\right).$$

Thus, the Lagrangian dual can be rewritten as (28). □

Now, we are able to prove the following result.

Theorem 4.1 *It holds that*

$$\begin{aligned} Conv_{f,X}(\mathbf{x}) = \max_{\mathbf{t},\boldsymbol{\gamma}}\ & h(\mathbf{t}) + \nabla h(\mathbf{t})(\mathbf{F}(\mathbf{x}) - \mathbf{t}) + \boldsymbol{\gamma}^\top \left[\mathbf{G}(\mathbf{t}) + \nabla \mathbf{G}(\mathbf{t})\left(\mathbf{F}(\mathbf{x}) - \mathbf{t}\right) - \mathbf{m}\right]. \\ & \nabla h(\mathbf{t}) + \boldsymbol{\gamma}^\top \nabla \mathbf{G}(\mathbf{t}) \geq \mathbf{0} \\ & \boldsymbol{\gamma} \geq \mathbf{0}. \end{aligned}$$

Proof When Φ_X is defined as in (27), we can rewrite (4) as

$$\begin{aligned} Conv_{f,X}(\mathbf{x}) = \max_{\boldsymbol{\alpha},\boldsymbol{\delta}\geq\mathbf{0}}\ \boldsymbol{\delta}^\top \mathbf{F}(\mathbf{x}) + \boldsymbol{\alpha}^\top (\mathbf{x} - \mathbf{x}^*) + \min\ & h(\mathbf{F}(\mathbf{y})) - \boldsymbol{\delta}^\top \mathbf{F}(\mathbf{y}) - \boldsymbol{\alpha}^\top (\mathbf{y} - \mathbf{x}^*) \\ & \mathbf{G}(\mathbf{F}(\mathbf{x})) \leq \mathbf{m}. \end{aligned}$$

In view of Proposition 4.1 we have, after eliminating $\boldsymbol{\delta}$,

$$\begin{aligned} Conv_{f,X}(\mathbf{x}) = \max \ & (\nabla h(\mathbf{t}) - \boldsymbol{\nu} + \boldsymbol{\gamma}^\top \nabla \mathbf{G}(\mathbf{t}))^\top \mathbf{F}(\mathbf{x}) + \boldsymbol{\gamma}^\top \mathbf{G}(\mathbf{t}) - \boldsymbol{\gamma}^\top \nabla \mathbf{G}(\mathbf{t})\mathbf{t} - \boldsymbol{\gamma}^\top \mathbf{m} \\ & + \boldsymbol{\alpha}^\top \mathbf{x} + \boldsymbol{\nu}^\top \mathbf{F}\left(\mathbf{F}'^{-1}\left(\tfrac{\boldsymbol{\alpha}}{\boldsymbol{\nu}}\right)\right) - \boldsymbol{\alpha}^\top \mathbf{F}'^{-1}\left(\tfrac{\boldsymbol{\alpha}}{\boldsymbol{\nu}}\right) + h(\mathbf{t}) - \nabla h(\mathbf{t})\mathbf{t} \\ & \nabla h(\mathbf{t}) - \boldsymbol{\nu} + \boldsymbol{\gamma}^\top \nabla \mathbf{G}(\mathbf{t}) \geq \mathbf{0} \\ & \boldsymbol{\nu}, \boldsymbol{\gamma} \geq \mathbf{0} \\ & \boldsymbol{\alpha} \in \mathbb{R}^n. \end{aligned}$$

After imposing the stationarity condition, it can be seen that the optimal solution of

$$\max_{\boldsymbol{\alpha}} \boldsymbol{\alpha}^\top \mathbf{x} + \boldsymbol{\nu}^\top \mathbf{F}\left(\mathbf{F}'^{-1}\left(\frac{\boldsymbol{\alpha}}{\boldsymbol{\nu}}\right)\right) - \boldsymbol{\alpha}^\top \mathbf{F}'^{-1}\left(\frac{\boldsymbol{\alpha}}{\boldsymbol{\nu}}\right),$$

is $\boldsymbol{\alpha} = \boldsymbol{\nu} \odot \mathbf{F}(\mathbf{x})$, where $\odot$ denotes the component-wise product. Then, after substitution, the optimal value of this maximization problem is $\boldsymbol{\nu}^\top \mathbf{F}(\mathbf{x})$, and, consequently, we have

$$\begin{aligned} Conv_{f,X}(\mathbf{x}) = \max_{\mathbf{t},\boldsymbol{\gamma},\boldsymbol{\nu}} \ & h(\mathbf{t}) + \nabla h(\mathbf{t})(\mathbf{F}(\mathbf{x}) - \mathbf{t}) + \boldsymbol{\gamma}^\top \left[\mathbf{G}(\mathbf{t}) + \nabla \mathbf{G}(\mathbf{t})\left(\mathbf{F}(\mathbf{x}) - \mathbf{t}\right) - \mathbf{m}\right]. \\ & \nabla h(\mathbf{t}) - \boldsymbol{\nu} + \boldsymbol{\gamma}^\top \nabla \mathbf{G}(\mathbf{t}) \geq \mathbf{0} \\ & \boldsymbol{\nu}, \boldsymbol{\gamma} \geq \mathbf{0}. \end{aligned}$$

The maximum is attained at $\boldsymbol{\nu} = \mathbf{0}$ which leads to the desired result. □

Different special cases, all derived from Theorem 4.1, are the following:

- h linear: If we assume that h is linear, i.e., $h(\mathbf{t}) = \boldsymbol{\mu}^\top \mathbf{t}$, then

  $$Conv_{f,X}(\mathbf{x}) = \boldsymbol{\mu}^\top \mathbf{F}(\mathbf{x}) + \max_{\boldsymbol{\gamma} \geq \mathbf{0},\ \boldsymbol{\mu} + \boldsymbol{\gamma}^\top \nabla \mathbf{G}(\mathbf{t}) \geq \mathbf{0}} \boldsymbol{\gamma}^\top \left[\mathbf{G}(\mathbf{t}) + \nabla \mathbf{G}(\mathbf{t})\left(\mathbf{F}(\mathbf{x}) - \mathbf{t}\right) - \mathbf{m}\right].$$

  If we further assume that $\mathbf{G}$ is linear, i.e., $\mathbf{G}(\mathbf{t}) = \mathbf{M}\mathbf{t}$, then

  $$Conv_{f,X}(\mathbf{x}) = \boldsymbol{\mu}^\top \mathbf{F}(\mathbf{x}) + \max_{\boldsymbol{\gamma} \geq \mathbf{0},\ \boldsymbol{\mu} + \boldsymbol{\gamma}^\top \mathbf{M} \geq \mathbf{0}} \boldsymbol{\gamma}^\top \left[\mathbf{M}\mathbf{F}(\mathbf{x}) - \mathbf{m}\right].$$

  Note that in this case the value of the convex envelope is given by the solution of a linear program.

- h quadratic: If we assume that h is quadratic, i.e., $h(\mathbf{t}) = \frac{1}{2}\mathbf{t}^\top \mathbf{A}\mathbf{t} + \boldsymbol{\mu}^\top \mathbf{t}$, where $\mathbf{A}$ is a positive semidefinite matrix, then

$$\begin{aligned} Conv_{f,X}(\mathbf{x}) = \max\ & -\tfrac{1}{2}\mathbf{t}^\top \mathbf{A}\mathbf{t} + (\mathbf{A}\mathbf{t} + \boldsymbol{\mu})^\top \mathbf{F}(\mathbf{x}) + \boldsymbol{\gamma}^\top \\ & [\mathbf{G}(\mathbf{t}) + \nabla \mathbf{G}(\mathbf{t})\,(\mathbf{F}(\mathbf{x}) - \mathbf{t}) - \mathbf{m}]\,. \\ & \mathbf{A}\mathbf{t} + \boldsymbol{\mu} + \boldsymbol{\gamma}^\top \nabla \mathbf{G}(\mathbf{t}) \geq \mathbf{0} \\ & \boldsymbol{\gamma} \geq \mathbf{0}. \end{aligned}$$

If we further assume that $\mathbf{G}$ is linear, i.e., $\mathbf{G}(\mathbf{t}) = \mathbf{M}\mathbf{t}$, then

$$\begin{aligned} Conv_{f,X}(\mathbf{x}) = \max\ & -\tfrac{1}{2}\mathbf{t}^\top \mathbf{A}\mathbf{t} + (\mathbf{A}\mathbf{t} + \boldsymbol{\mu})^\top \mathbf{F}(\mathbf{x}) + \boldsymbol{\gamma}^\top\,[\mathbf{M}\mathbf{F}(\mathbf{x}) - \mathbf{m}]\,. \\ & \mathbf{A}\mathbf{t} + \boldsymbol{\mu} + \boldsymbol{\gamma}^\top \mathbf{M} \geq \mathbf{0} \\ & \boldsymbol{\gamma} \geq \mathbf{0}. \end{aligned}$$

A possible example is the following.

Example 4.1 A subcase with h linear and $\mathbf{G}$ nonlinear is the following. Let $k = 1$ and

$$\begin{aligned} & f_i(x_i) = |x_i|^p, \quad p \in (1, 2] \\ & g_1(\mathbf{t}) = \textstyle\sum_{i=1}^n t_i^{\frac{2}{p}}, \quad m_1 = 1, \end{aligned}$$

so that function f is $\sum_{i=1}^n \mu_i |x_i|^p$, while the region X is the unit sphere. Note, in particular, that for $\mu_i = -1$, $i = 1, \ldots, n$, function f is the opposite of the p-norm. Let

$$I_+ = \{i : \ \mu_i \geq 0\}, \quad I_- = \{1, \ldots, n\} \backslash I_+.$$

We assume that $I_- = \{i_1, \ldots, i_\ell\}$ and that its elements are ordered in such a way that for $r = 1, \ldots, \ell - 1$,

$$\frac{-p\mu_{i_r}}{2|x_{i_r}|^{2-p}} \leq \frac{-p\mu_{i_{r+1}}}{2|x_{i_{r+1}}|^{2-p}}.$$

It follows from Theorem 4.1 that

$$\begin{aligned} Conv_{f,X}(\mathbf{x}) = \textstyle\sum_{i=1}^n \mu_i |x_i|^p + \max\ & \gamma\,(g_1(\mathbf{t}) + \nabla g_1(\mathbf{t})(\mathbf{F}(\mathbf{x}) - \mathbf{t}) - 1) \\ & \mu_i + \gamma \tfrac{2}{p} |t_i|^{\frac{2-p}{p}} \geq 0 \qquad i = 1, \ldots, n \\ & \gamma \geq 0. \end{aligned}$$

The function to be maximized with respect to $\gamma \geq 0$ is subdivided into the intervals $\left[\frac{-p\mu_{i_r}}{2|x_{i_r}|^{2-p}}, \frac{-p\mu_{i_{r+1}}}{2|x_{i_{r+1}}|^{2-p}}\right]$, $r = 0, \ldots, \ell$, where the left extreme of the first interval is set equal to 0, while the right extreme of the last interval is set equal to $+\infty$. Over the rth interval the function is equal to

$$\gamma\left[\sum_{i\in I_+} x_i^2 + \sum_{s=1}^{r} x_{i_s}^2 - 1\right] + \gamma^{-\frac{p}{2-p}} \sum_{s=r+1}^{\ell}\left[\frac{(p-2)}{p}\left(\frac{-p\mu_{i_s}}{2}\right)^{\frac{2}{2-p}}\right] - \sum_{s=r+1}^{\ell} \mu_{i_s}|x_{i_s}|^p.$$

The derivative of this piecewise defined function tends to $+\infty$ for $\gamma \to 0$ and, for $\gamma \geq 0$, it is decreasing until it becomes constant and non positive (in fact, negative if $\mathbf{x}$ belongs to the interior of the unit sphere) for $\gamma \geq \frac{-p\mu_{i_\ell}}{2|x_{i_\ell}|^{2-p}}$. Thus, the function is concave for $\gamma \geq 0$ and has a unique maximizer within the interval $\left[\frac{-p\mu_{i_r}}{2|x_{i_r}|^{2-p}}, \frac{-p\mu_{i_{r+1}}}{2|x_{i_{r+1}}|^{2-p}}\right]$ such that

$$\sum_{i\in I_+} x_i^2 + \sum_{s=1}^{r} x_{i_s}^2 + x_{i_r}^2 \sum_{s=r+1}^{\ell}\left(\frac{\mu_{i_s}}{\mu_{i_r}}\right)^{\frac{2}{2-p}} - 1 \geq 0$$

and

$$\sum_{i\in I_+} x_i^2 + \sum_{s=1}^{r+1} x_{i_s}^2 + x_{i_{r+1}}^2 \sum_{s=r+2}^{\ell}\left(\frac{\mu_{i_s}}{\mu_{i_{r+1}}}\right)^{\frac{2}{2-p}} - 1 \leq 0,$$

where the two values above are the derivative values at the two extremes of the interval.

5 Conclusions and future work

The value of the convex envelope of a function f over a region X at some point $\mathbf{x} \in X$ is the largest value at $\mathbf{x}$ of affine understimating functions for f over X. This value can be computed by solving a (usually hard) maximin problem. Computing convex envelopes or, at least, good convex underestimators, may enhance the quality of lower bounds for nonconvex optimization problems and, thus, the performance of global optimization codes for their solution. In this paper we have shown that good convex underestimating functions can be efficiently derived by the following scheme: (i) enlarge the class of convex underestimating functions (not only affine ones); (ii) relax the inner minimization problem; (iii) consider the dual of the relaxation. In particular, when the relaxation is exact and when strong duality holds, the value computed is the one of the convex envelope. In the paper the scheme above is applied to different problems, in particular quadratic problems with quadratic constraints, and different cases for which the resulting underestimating function is the convex envelope are recognized. Some results are also discussed for polynomial and ratio of polynomial functions over regions defined by polynomials, and for some separable functions over regions defined by similarly defined separable functions. As possible topics for future research, we would like to explore further classes of problems where the proposed technique can be applied, and to employ the convex underestimators derived through it within the framework of global optimization codes.

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Conflict of interest The author has no competing interest to declare that are relevant to the content of this article.

References

1. Maranas, C.D., Floudas, C.A.: Global minimum potential energy conformations of small molecules. J. Glob. Optim. **4**, 135–170 (1994)
2. Meyer, C.A., Floudas, C.A.: Convex underestimation of twice continuously differentiable functions by piecewise quadratic perturbation: spline αBB underestimators. J. Glob. Optim. **32**, 221–258 (2005)
3. Akrotirianakis, I.G., Floudas, C.A.: A new class of improved convex underestimators for twice continuously differentiable constrained NLPs. J. Glob. Optim. **30**, 367–390 (2004)
4. Zhu, Y., Kuno, T.: A global optimization method, QBB, for twice-differentiable nonconvex optimization problem. J. Glob. Optim. **33**, 435–464 (2005)
5. Bompadre, A., Mitsos, A.: Convergence rate of McCormick relaxations. J. Glob. Optim. **52**, 1–28 (2012)
6. Scott, J.K., Stuber, M.D., Barton, P.I.: Generalized McCormick relaxations. J. Glob. Optim. **51**, 569–606 (2011)
7. Meyer, C.A., Floudas, C.A.: Convex envelopes for edge-concave functions. Math. Program. **103**, 207–224 (2005)
8. Rikun, A.: A convex envelope formula for multilinear functions. J. Glob. Optim. **10**, 425–437 (1997)
9. Tardella, F.: On the existence of polyhedral convex envelopes. In: Floudas, C.A., Pardalos, P.M. (eds.) Frontiers in Global Optimization, pp. 563–574. Kluwer Academic Publishers, Dordrecht (2003)
10. Tardella, F.: Existence and sum decomposition of vertex polyhedral convex envelopes. Optim. Lett. **2**, 363–375 (2008)
11. De Rosa, A., Khajavirad, A.: Explicit convex hull description of bivariate quadratic sets with indicator variables. https://arxiv.org/pdf/2208.08703.pdf
12. Jach, M., Michaels, D., Weismantel, R.: The convex envelope of (n-1)-convex functions. SIAM J. Optim. **19**(3), 1451–1466 (2008)
13. Khajavirad, A., Sahinidis, N.V.: Convex envelopes of products of convex and component-wise concave functions. J. Glob. Optim. **51**, 391–409 (2012)

14. Khajavirad, A., Sahinidis, N.V.: Convex envelopes generated from finitely many compact convex sets. Math. Program. **137**, 371–408 (2013)
15. Locatelli, M.: Non polyhedral convex envelopes for 1-convex functions. J. Glob. Optim. **65**, 637–655 (2016)
16. Locatelli, M.: Convex envelopes of bivariate functions through the solution of KKT systems. J. Glob. Optim. **72**, 277–303 (2018)
17. Nguyen, T.T., Richard, J.-P.P., Tawarmalani, M.: Deriving convex hulls through lifting and projection. Math. Program. **169**, 377–415 (2018)
18. Tawarmalani, M., Richard, J.-P.P., Xiong, C.: Explicit convex and concave envelopes through polyhedral subdivisions. Math. Program. **138**, 531–577 (2013)
19. Matsui, T.: NP-hardness of linear multiplicative programming and related problems. J. Glob. Optim. **9**, 113–119 (1996)
20. Ben-Ameur, W., Ouorou, A., Wang, G.: Convex and concave envelopes: revisited and new perspectives. Oper. Res. Lett. **45**(5), 421–426 (2017)
21. Kim, S., Kojima, M.: Exact solutions of some nonconvex quadratic optimization problems via SDP and SOCP relaxations. Comput. Optim. Appl. **26**, 143–154 (2003)
22. Ben-Tal, A., Den Hertog, D.: Hidden conic quadratic representation of some nonconvex quadratic optimization problems. Math. Program. **143**, 1–29 (2014)
23. Wang, A.L., Kilinç-Karzan, F.: The generalized trust region subproblem: solution complexity and convex hull results. Math. Program. **191**, 445–486 (2022)
24. Hsia, Y., Sheu, R.-L.: Trust region subproblem with a fixed number of additional linear inequality constraints has polynomial complexity. http://arxiv.org/abs/1312.1398
25. Jeyakumar, V., Li, G.Y.: Trust-region problems with linear inequality constraints: exact SDP relaxation, global optimality and robust optimization. Math. Program. **147**, 171–206 (2014)
26. Beck, A., Eldar, Y.C.: Strong duality in nonconvex quadratic optimization with two quadratic constraints. SIAM J. Optim. **17**(3), 844–860 (2006)
27. Laraki, R., Lasserre, J.B.: Computing uniform convex approximations for convex envelopes and convex hulls. J. Convex Anal. **15**(3), 635–654 (2008)
28. Ahmadi, A.A., Olshevsky, A., Parrilo, P.A., Tsitsiklis, J.N.: NP-hardness of deciding convexity of quartic polynomials and related problems. Math. Program. **137**, 453–476 (2013)
29. Helton, J.W., Nie, J.: Semidefinite representation of convex sets. Math. Program. **122**, 21–64 (2010)
30. Lasserre, J.B., Thanh, T.P.: Convex underestimators of polynomials. J. Glob. Optim. **56**, 1–25 (2013)
31. Ben-Tal, A., Den Hertog, D., Laurent, M.: Hidden Convexity in Partially Separable Optimization. Technical Report 2011–70. Tilburg University, Center for Economic Research (2011)