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COAP 2009 best paper award

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In each year, the Computational Optimization and Applications (COAP) editorial board selects a paper from the preceding year's COAP publications for the "Best Paper Award." The recipients of the award for papers published in 2009 are Samuel Burer of the University of Iowa and Dieter Vandenbussche of Axioma Inc. for their paper "Globally solving box-constrained nonconvex quadratic programs with semidefinite-based finite branch-and-bound," published in Volume 43 pages 181–195.

This paper describes work done by the authors during the years 2002–2007. It was part of a collaboration mostly motivated by the desire of each author to gain a greater understanding of the other's field of expertise. One author, Samuel Burer, had worked mainly in semidefinite programming (SDP), while the other, Dieter Vandenbussche, had worked mostly in the area of applying integer programming (IP) techniques to solve continuous, non-convex optimization problems. Both authors were intrigued by the large volume of literature describing the use of SDP relaxations for obtaining improved bounds for 0-1 IP problems, much of which can be traced to the seminal paper by Lovász and Schrijver [8]. However, the literature dealt mainly with the theory of these relaxations, and it was clear that, for realistic IPs, the large size of the proposed SDP relaxations posed a serious computational challenge for existing SDP algorithms. So the authors focused on developing specialized algorithms for solving, possibly approximately, these large, structured SDPs. There was already some precedent for this type of effort; see for example the survey paper by Monteiro [9], which discusses first-order methods for large-scale SDPs.

This initial collaboration resulted in an algorithm [6] combining an augmented Lagrangian algorithm with block coordinate descent to obtain an approximate solution of very large SDPs obtained from relaxing 0-1 IP problems. The idea behind the algorithm was to decompose the problem by penalizing certain constraints in the objective and then to find optimal dual multipliers using a classic augmented Lagrangian approach. Each iteration required the solution of a large SDP with a quadratic objective. However, rather than solving this subproblem exactly, the authors used block coordinate descent, which decomposed each iteration into the solution of several independent convex QPs as well as the projection of a matrix into the cone of positive semidefinite matrices, which amounts to a spectral decomposition. Somewhat surprisingly, computational experiments showed that the dual multipliers converged reasonably well even when the subproblems were solved very loosely by only a few cycles of block coordinate descent. This was sufficient, for example, to compute best-known bounds for several unsolved problems in the Quadratic Assignment Problem Library [11].

For globally optimizing linear IPs, the authors began to realize that, even with better techniques for the SDP relaxations, these relaxations could still be ineffective, either because the bounds might still be too weak on a given problem class, or because solving the SDPs, even approximately, might still be more time-consuming than a standard IP solver, which can evaluate many nodes of a branch-and-bound tree by quickly reoptimizing linear programming (LP) relaxations using the dual simplex method.

Hence, the next goal of the collaboration was to identify problem classes for which these SDPs relaxations would provide uniformly good bounds in a short amount of time. The authors also wanted to apply the relaxations in a branch-and-bound framework.

So the authors shifted their focus away from IPs to continuous problems in the form of non-convex quadratic programs (QPs), i.e., NP-hard problems with a non-convex quadratic objective and linear constraints. The motivation for looking at this class of problems was two-fold:

1. These problems admit SDP relaxations of very similar structure as the ones for IP, and a significant amount of literature indicated the strength of such SDP relaxations in the continuous case.
2. Vandenbussche and Nemhauser [13, 14] had developed a finite, LP-based branch-and-cut algorithm for solving non-convex QPs whose only constraints were bounds on the variables. This approach was based on reformulating the QP in terms of its KKT conditions, resulting in an LP with complementarity constraints. After this reformulation, Vandenbussche and Nemhauser applied several IP technologies with the end result that they could globally solve this problem class much faster than existing continuous techniques such as spatial branch-and-bound.

The authors wanted to combine the strengths of these two approaches—the first providing very strong bounds and the second providing a finite branch-and-bound scheme.

The first challenge was to generalize the work of Vandenbussche and Nemhauser [13] beyond bound constraints to general linear constraints. It was easy to reformulate the problem as an LP with complementarity constraints and to set up a finite branch-and-bound framework. The hard part was that the resulting LP and SDP relaxations were unbounded. However, by combining features of both relaxation types, the authors overcame this hurdle, thus obtaining relaxations that could be used in a finite branch-and-bound scheme. By solving the relaxations with the augmented Lagrangian algorithm developed previously, the authors were able to globally solve non-convex QPs quite reliably. The results of this work on non-convex QP can be found in Burer and Vandenbussche [7].

This brought the authors to their current *COAP* paper. They wanted to compare their SDP-based branch-and-bound algorithm with that of an LP-only approach, which led to a comparison with Vandenbussche and Nemhauser [13] on box-constrained non-convex QPs. In the paper, the authors show how to improve further their general SDP method by tailoring it to this specific class of problems. This resulted in an algorithm that, for relatively large instances of box-constrained QP (up to 100 variables), far outperformed the LP-based branch-and-cut method of Vandenbussche and Nemhauser [13].

While this research is based on several different ideas and contains many details, at its most basic level, it demonstrates the practical power of SDP relaxations for continuous, non-convex problems. To the authors knowledge, this is one of the first clear examples in which an SDP-based branch-and-bound algorithm outperforms an LP-based one.

More generally, the paper also illustrates that it can be worthwhile to pay for a computationally expensive, but strong, relaxation compared to evaluating many inexpensive relaxations. Furthermore, it shows that simple, classic ideas such as augmented Lagrangian and block coordinate descent still have an important role to play in modern, large-scale optimization.

With his current employer, [2], Vandenbussche employs the principles of using strong, though potentially computationally more expensive, relaxations to solve challenging portfolio construction optimization problems. In this case, the relaxations are in the form of Second Order Cone Programs (SOCPs). Regardless, the optimization team at Axioma has found that by paying a higher computational price for solving stronger relaxations, we obtain better primal solutions—and better bounds to convince clients of the quality of those primal solutions.

Since the authors completed this paper and the aforementioned related projects, similar ideas have appeared in others' research. For example, Povh et al. [10], Wen et al. [15], Zhao et al. [16], Burer [5] have each employed the augmented-Lagrangian idea with block coordinate descent (also called *alternating projections*) for solving large-scale SDPs. In addition, the box-constrained case of non-convex QP has become a standard test case in mixed integer nonlinear programming (MINLP), e.g., see [1, 3, 4, 12].

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