On-line appendix to the paper "Solving the Linear Multiple Choice Knapsack Problem with Two Objectives: Profit and Equity"

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Proof of Proposition 2

For clarity, we treat each option separately.

Option A: Assume that the application of Option A is stopped at some point before an associated stopping condition is encountered. At that point, the increasing and decreasing slopes of all sets whose cost is not modified by the application of this option remain unchanged. Moreover, the upper sets remain upper, their decreasing slopes remain unchanged, and the increasing slope of the increasing set remains unchanged, too. Therefore, the $\frac{\Delta P}{\Delta f}$ ratios of Options A and D remain unchanged, too. Since Option A was selected for application at the beginning of the iteration, its ratio was not greater than the ratio of Option C, i.e.,

$$a - mo \leq \frac{an - em}{m + n} \implies am + an - m^2 o - mno \leq an - em \implies a + e \leq mo + no \implies o \geq \frac{a + e}{m + n}. \quad (1)$$

In order to prove that the $\frac{\Delta P}{\Delta f}$ ratio of Option B can not be smaller than the ratio of Option A at the current solution, we consider two cases. The first is when the increasing set of Option A qualifies as decreasing for Option B. Since none of the stopping conditions of Option A has been encountered, the partial solution of this set corresponds to a point different than the two endpoints of the line segment connecting the two adjacent variables whose values are being changed in the associated upper hull. Therefore, the decreasing slope of the increasing set is equal to its increasing slope, o. If the $\frac{\Delta P}{\Delta f}$ ratio of Option B hasn't changed since the beginning of the iteration, it is still greater or equal to the ratio of Option A. If it has, we consider two subcases. The first is when the increasing set of Option A was internal at the beginning of the iteration. This, together with the fact that the ratio of Option B has changed, implies that the increasing set of Option A has different decreasing slope now and qualifies as decreasing for Option B. Setting the new ratio of Option B smaller than the ratio of Option A, we get: $a - mo > nu' - e$. Since the decreasing slope of the increasing set of Option A is equal to its increasing slope, we have $u' = o$ and the previous inequality becomes $a - mo > no - e \Longrightarrow o < \frac{a+e}{m+n}$. The second subcase is when the increasing set of Option A was lower at the beginning of the iteration. In this subcase, the number of lower sets drops from n to $n-1$, as soon as the cost of this set increases from its initial value. Since the increasing set of Option A qualifies as decreasing for Option B, the new $\frac{\Delta P}{\Delta f}$ ratio of Option B is equal to $u'(n-1) - e'$. Since $u' = o$ and $e' = e - o$, this expression is equal to $o(n-1) - (e-o) = no - e$, and again, $o < \frac{a+e}{m+n}$ needs to hold in order for Option's B ratio to be smaller than the ratio of Option A. This inequality however, contradicts inequality (1); therefore the ratio of Option B can not be smaller than the ratio of Option A at the current solution. The second case is when the increasing set of Option A does not qualify as decreasing for Option B. In this case, the non-lower set with the minimum decreasing slope is still the same. If the increasing set of Option A was internal at the beginning of the iteration, then the $\frac{\Delta P}{\Delta f}$ ratio of Option B is still the same. If it was lower, the new ratio of Option B is equal to $u(n-1) - e' = u(n-1) - (e-o) = nu - e + o - u$. Note however, that $o \geq u$, otherwise the increasing set of Option A would be selected as decreasing for Option B. Thus, we have $nu - e + o - u \geq nu - e$; hence, the ratio of Option B can not be smaller than what it was at the beginning of the iteration. Therefore, the ratio of Option B can not be smaller than the ratio of Option A at the current solution.

Consider now Options C and E. If the increasing set of Option A was internal at the beginning of the iteration, then clearly the $\frac{\Delta P}{\Delta f}$ ratios of Options C and E remain unchanged at the current solution. Assume now that the increasing set was lower at the beginning of the iteration. In this case, the number of lower sets drops from n to $n-1$, as soon as the cost of this set increases from its initial value. Its increasing slope must be nonnegative, otherwise Option D would have been selected over Option A at the beginning of the iteration. Therefore, the new sum of increasing slopes of the lower sets at the current solution is $e' \leq e$, which means that $-e' \geq -e$. Hence, the new $\frac{\Delta P}{\Delta f}$ ratio of Option E is not smaller than before. Additionally, the new $\frac{\Delta P}{\Delta f}$ ratio of Option C is equal to $\frac{a(n-1)-e'm}{m+n-1} = \frac{a(n-1)-m(e-o)}{m+n-1}$. In order for Option's C ratio to be smaller than the ratio of Option A at the current solution, we must have $a - mo > \frac{a(n-1)-m(e-o)}{m+n-1} \Longrightarrow am + an - a - m^2o - mno + mo > an - a - em + mo \Longrightarrow a - mo - no > -e \Longrightarrow o < \frac{a+e}{m+n}$. This inequality however is in contradiction with inequality (1); therefore the new ratio of Option C can not be smaller than the ratio of Option A. Finally, note that the resource residual hasn't changed since the beginning of the iteration. Therefore, the situation in which the application of Option E was not feasible at the beginning of the iteration, but is now, can not come up. This completes the proof of the proposition for the case that the applied option is A.

Option B: Assume that the application of Option B is stopped at some point before an associated stopping condition is encountered. In this case, it is easy to see that the $\frac{\Delta P}{\Delta f}$ ratios of Options B and E remain unchanged. Using the fact that Option B was selected over Option C at the beginning of the iteration, we get

$$u \leq \frac{a+e}{m+n}. \tag{2}$$

The proof that Option A can not have smaller $\frac{\Delta P}{\Delta f}$ ratio when Option B is applied is symmetric to the proof that Option B can not have smaller $\frac{\Delta P}{\Delta f}$ ratio when Option A is applied, and is not repeated here, for space consideration. Consider now Options C and D. If the decreasing set of Option B was internal at the beginning of the iteration, then clearly the $\frac{\Delta P}{\Delta f}$ ratios of Options C and D remain unchanged at the current solution. Assume now that the decreasing set was upper at the beginning of the iteration. In this case, the number of upper sets drops from m to $m-1$, as soon as the cost of this set decreases from its initial value. Since the value of the first objective was the maximum possible for the value of the second objective at the beginning of the iteration, Lemma 2, introduced in Section 2.3, holds; therefore, the decreasing slope of the decreasing set, u, must be nonnegative. The sum of decreasing slopes of the upper sets at the current solution is $a' = a - u$. In order for Option's D ratio to be smaller than the ratio of Option B, we must have $nu - e > a - u \Longrightarrow u(n+1) > a + e \Longrightarrow u > \frac{a+e}{n+1}$. This inequality however, contradicts inequality (2), since u is nonnegative, which implies that $a+e$ is nonnegative, too; therefore, $\frac{a+e}{n+1} \geq \frac{a+e}{m+n}$. Therefore, the new ratio of Option D can not be smaller than the ratio of Option B. Finally, the new $\frac{\Delta P}{\Delta f}$ ratio of Option C is equal to $\frac{a'n-e(m-1)}{m+n-1} = \frac{n(a-u)-e(m-1)}{m+n-1}$. In order for Option's C ratio to be smaller than the ratio of Option B at the current solution, we must have $nu - e > \frac{n(a-u)-e(m-1)}{m+n-1}$ $\Longrightarrow mnu + n^2u - nu - em - en + e > an - nu - em + e \Longrightarrow mu + nu - e > a \Longrightarrow u > \frac{a+e}{m+n}$. This inequality however, contradicts inequality (2); therefore, the new ratio of Option C can not be smaller than the ratio of Option B. Finally, note that the resource residual hasn't changed since the beginning of the iteration. Therefore, the situation in which the application of Option E was not feasible at the beginning of the iteration, but is now, can not come up. This completes the proof of the proposition for the case that the applied option is B.

Option C: Assume that the application of Option C is stopped at some point before an associated stopping condition is encountered. At that point, the increasing and decreasing slopes of all sets whose cost is not modified by the application of this option remain unchanged. Moreover, the lower sets remain lower, their increasing slopes remain unchanged, the upper sets remain upper and their decreasing slopes remain unchanged, too. Therefore, the $\frac{\Delta P}{\Delta f}$ ratios of Options A, B, C, D and E remain unchanged, too, which means that the ratio of Option C is still the same. Finally, note that the resource residual hasn't changed since the beginning of the iteration. Therefore, the situation in which the application of Option E was not feasible at the beginning of the iteration, but is now, can not come up. This completes the proof of the proposition for the case that the applied option is C.

Option D: Assume that the application of Option D is stopped at some point before an associated stopping condition is encountered. At that point, the increasing and decreasing slopes of all sets whose cost is not modified by the application of the option remain unchanged. Moreover, the upper sets remain upper and their decreasing slopes remain unchanged. Therefore, the $\frac{\Delta P}{\Delta f}$ ratios of Options A, B, C, D and E remain unchanged, too. Note additionally, that since Option D was selected at the beginning of the iteration over Option C, we have $a \leq \frac{an-em}{m+n} \Longrightarrow am + an \leq an - em \Longrightarrow a \leq -e$. Therefore, the case in which Option E had smaller $\frac{\Delta P}{\Delta f}$ ratio since the beginning of the iteration but was not applied because the resource residual was not positive can not come up. This completes the proof of the proposition for the case that the applied option is D.

Option E: Assume that the application of Option E is stopped at some point before an associated stopping condition is encountered. At that point, the increasing and decreasing slopes of all sets whose cost is not modified by the application of the option remain unchanged. Moreover, the lower sets remain lower and their increasing slopes remain unchanged. Therefore, the $\frac{\Delta P}{\Delta f}$ ratios of Options A, B, C, D and E remain unchanged, too, which means that the ratio of Option E is still the same. This completes the proof of the proposition for the case that the applied option is E.

□

Proof of Lemma 1

Consider any LMCKE instance and assume that the interval $[L, U]$ containing the optimal costs of the sets is known. For reasons of clarity, we call this the "original" instance. Assume now that we want to solve the original LMCKE instance for a new value $f' < f$, where $\Delta f = f' - f$ is arbitrarily close to 0. We call this the "modified" instance. Since the new interval $[L', U']$ must have width which is marginally smaller than the width of the previous interval, at least one of the following must hold: 1) $U' < U$, or 2) $L' > L$. Therefore, we examine what happens when the endpoints of the interval containing the optimal costs of the sets of the original LMCKE instance are slightly perturbed. Consider the associated LMCK instance with lower bound L'' and upper bound $U'' = L'' + f'$ on the cost of each set, where L'' is marginally smaller than L. The solution procedure allocates initially a resource amount equal to L'' to each set. The order of the decision variables in the multiple choice lists and the master list is the same as in the original instance. We consider two subcases. If the resource residual at the optimal solution of the original LMCKE instance is positive, then the increasing slope of every non-upper set in that solution is nonpositive. Therefore, during the allocation of the remaining resource in the modified LMCKE instance, we get an intermediate solution, in which the cost of some of the originally lower sets is equal to L'', the cost of the remaining originally lower sets is equal to L, the cost of the originally internal sets is the same as before, and the cost of the originally upper sets is equal to U''. For the originally lower sets which are lower in this solution, this is due to the fact that their increasing slope is not positive when their cost is equal to L''. For the originally lower sets which are not lower in this solution, this is due to the fact that their increasing slope is positive when their cost is equal to L'', but nonpositive when their cost is equal to L. For the originally upper sets, this is due to the fact that the upper bound is now U'', instead of U. On the other hand, the originally internal sets are not affected by the change on the two bounds; therefore, their costs assume their original values. Hence, every non-upper set has nonpositive increasing slope, and the algorithm terminates, which means that this solution is optimal for the associated LMCK instance with bounds L'' and U''. If the resource residual at the optimal solution of the original LMCKE instance is equal to 0, consider the non-upper set in that solution with the most positive increasing slope (if such a set exists, we call it the increasing set). During the allocation of the remaining resource in the modified LMCKE instance, we get an intermediate solution, in which the cost of every set is the same as in the optimal solution of the first subcase. The resource residual is strictly positive in this solution; therefore, it can be used to increase the cost of the increasing set. Since Δf is arbitrarily close to 0, the resource residual drops to 0 before the increasing slope of the increasing set changes. Then, the algorithm terminates and this solution is optimal for the LMCK instance with bounds L'' and U''. At the optimal solution obtained in either of the two subcases, let n_1 be the number of originally lower sets whose cost is equal to L'' and $\Delta L = L'' - L$. Then, the number of lower sets is n_1 and the number of upper sets m, where m is the number of upper sets at the optimal solution of the original LMCKE instance. With respect to the optimal solution of the original LMCKE instance, the cost of the upper sets drops by $|\Delta f| + |\Delta L|$, the cost of the n_1 lower sets drops by $|\Delta L|$, the cost of the increasing set (if it exists) increases by $m(|\Delta f| + |\Delta L|) + n_1 |\Delta L|$, and every other set has the same cost as before. Let o be the increasing slope of the increasing set, a be the sum of decreasing slopes of the upper sets and g be the sum of decreasing slopes of the n_1 lower sets. The difference in total profit with respect to the optimal solution of the original LMCKE instance is equal to $-a(|\Delta f| + |\Delta L|) - g|\Delta L| + o(m(|\Delta f| + |\Delta L|) + n_1 |\Delta L|) = (a - mo)\Delta f + (a + g - mo - n_1 o)\Delta L$. Therefore, $\frac{\Delta P}{\Delta f} = a - mo + \frac{(a+g-mo-n_1 o)\Delta L}{\Delta f}$, where $o \geq 0$. This expression holds when the increasing set does not exist, too, since we can set $o = 0$ in that case. If this ratio is smaller than the ratio obtained when Option A with the same increasing set is applied to the optimal solution of the original LMCKE instance, we have $a - om + \frac{(a+g-mo-n_1 o)\Delta L}{\Delta f} < a - om \iff \frac{(a+g-mo-n_1 o)\Delta L}{\Delta f} < 0$. Note that, since ΔL and Δf are both negative, this expression implies that the term $(a + g - mo - n_1 o)$ is negative, too. Suppose that at the optimal solution of the original LMCKE instance we decrease the cost of the upper and the same n_1 lower sets by $|\Delta L|$ and we allocate the recovered resource amount to the same increasing set. The solution that we get is feasible for the original LMCKE instance, and the resulting change in total profit is equal to $o(m+n_1)|\Delta L| - (a+g)|\Delta L| = -(a+g-mo-n_1 o)|\Delta L|$, which due to the above is strictly positive. This however is a contradiction, since the initial solution is optimal for the original instance, which implies that we can not move to another feasible solution with higher profit. Therefore, taking $L'' < L$ does not lead to a marginal decrease in P per unit decrease in f which is smaller than the one that can be achieved by taking $L'' \geq L$. Consider now the associated LMCK instance with upper bound U'' and lower bound $L'' = U'' - f'$ on the cost of each set, where U'' is marginally larger than U. The solution procedure allocates initially a resource amount equal to L'' to each set. The order of the decision variables in the multiple choice lists and the master list is the same as in the original instance. Assuming that the remaining resource is enough, during its allocation in the modified LMCKE instance, we get an intermediate solution, in which the cost of the originally lower sets is equal to L'', the cost of the originally internal sets is the same as before, the cost of some of the originally upper sets is equal to U and the cost of the remaining originally upper sets is equal to U''. For the originally lower sets, this is due to the fact that the lower bound is now L'', instead of L. For the originally upper sets which are not upper in this solution, this is due to the fact that their increasing slope is nonpositive when their cost is equal to U. For the originally upper sets which are upper in this solution, this is due to the fact that their increasing slope is positive when their cost is equal to U, but their cost can not be increased beyond U'', due to the upper bound constraint. On the other hand, the originally internal sets are not affected by the change on the two bounds; therefore their costs assume their original values. Note that since the cost of the lower sets now increases to L'' and the cost of some of the originally upper sets increases to U'', an extra resource amount is required for this solution. We consider two subcases. If the resource residual at the optimal solution of the original LMCKE instance is positive, this amount is available. Then, the algorithm terminates, due to the fact that every non-upper set has nonpositive increasing slope, since it had nonpositive increasing slope with same or smaller cost at the optimal solution of the original LMCKE instance. If not, then the cost of the non-lower set which is increased last (if such a set exists, we call it the decreasing set) will be smaller than before by this amount. Then, the algorithm terminates and the current solution is optimal for the associated LMCK instance with bounds L'' and U''. At the solution obtained in either of the two subcases, let m_1 be the number of originally upper sets whose cost is now equal to U'' and $\Delta U = U'' - U$. The number of upper sets in this solution is m_1 and the number of lower sets n, where n is the number of lower sets at the optimal solution of the original LMCKE instance. With respect to the optimal solution of the original LMCKE instance, the cost of the lower sets increases by $|\Delta f| + \Delta U$, the cost of the m_1 upper sets increases by ΔU, the cost of the decreasing set (if it exists) decreases by $n(|\Delta f| + \Delta U) + m_1\Delta U$, and every other set has the same cost as before. Let u be the decreasing slope of the decreasing set, e be the sum of increasing slopes of the lower sets and g be the sum of increasing slopes of the m_2 upper sets. Following the same reasoning as Lemma 2, introduced in Section 2.3, we can easily prove that $u \geq 0$. The difference in total profit with respect to the optimal solution of the original LMCKE instance is equal to $e(|\Delta f|+\Delta U)+g\Delta U-u(n(|\Delta f|+\Delta U)+m_1\Delta U) = (nu-e)\Delta f+(e+g-nu-m_1u)\Delta U$; therefore, $\frac{\Delta P}{\Delta f} = nu-e+\frac{(e+g-nu-m_1u)\Delta U}{\Delta f}$. This expression holds when the decreasing set does not exist, too, since we can set $u = 0$ in that case. If this ratio is smaller than the ratio obtained when Option B with the same decreasing set is applied to the optimal solution of the original LMCKE instance, we have $nu - e + \frac{(e+g-nu-m_1u)\Delta U}{\Delta f} < nu - e \iff \frac{(e+g-nu-m_1u)\Delta U}{\Delta f} < 0$. Note that since ΔU is positive and Δf negative, this expression implies that the term $(e + g - nu - m_1u)$ is positive. Suppose that at the optimal solution of the original LMCKE instance, we increase the cost of the same m_1 upper and the lower sets by ΔU and we decrease the cost of the same decreasing set by $(n + m_1)\Delta U$. Note that the solution that we get is feasible and the resulting change in total profit is equal to $(e + g)\Delta U - u(n + m_1)\Delta U = (e + g - nu - m_1u)\Delta U$, which due to the above is strictly positive. This however is a contradiction, since the initial solution is optimal for the original LMCKE instance, which means that we can not move to another feasible solution with higher profit. Therefore, neither taking $U'' > U$ leads to a marginal decrease in P per unit decrease in f which is smaller than the one that can be achieved by taking $U'' \leq U$. This completes the proof of the lemma. □

Proof of Proposition 3

Consider any LMCKE instance. For reasons of clarity, we call this the "original" instance. Assume that the interval $[L, U]$ containing the optimal costs of the sets is known, and without loss of generality that L = mincost, U = maxcost and $f = U - L$. Assume also that we want to solve the original LMCKE instance for a new value $f' < f$, where $\Delta f = f' - f$ is arbitrarily close to 0. We call this the "modified" instance. Since the new interval $[L', U']$ must have width which is marginally smaller than the width of the previous interval, and additionally, $L' \geq L$ and $U' \leq U$ from Lemma 1, we distinguish three cases: 1) $L' = L$ and $U' < U$, 2) $L' > L$ and $U' = U$, and 3) $L' > L$ and $U' < U$. If case 1 is true, consider the associated LMCK instance, after adding the constraints $C_k \geq L$ and $C_k \leq U'$, for all $k \in S$. The order of the decision variables in the multiple choice lists and the master list is the same as in the original instance. The solution procedure allocates initially a resource amount equal to L to each set. During the allocation of the remaining resource, we get an intermediate solution in which the cost of every non-upper set is the same as before, and the cost of every upper set is equal to U'. The number of upper sets in this solution is equal to the number of upper sets at the optimal solution of the original LMCKE instance. If this number is m, then the resource residual of the current solution is larger than the resource residual of that solution by $m\,|\Delta f|$. We distinguish two subcases. If the resource residual at the optimal solution of the original LMCKE instance is positive, then the increasing slope of every non-upper set in that solution is nonpositive. Since the cost of every non-upper set at the current solution is the same as in that solution, the algorithm terminates. Clearly, this is equivalent to the application of Option D. If the resource residual at the optimal solution of the original LMCKE instance is equal to 0, then there may exist a non-upper set at the current solution with positive increasing slope. In this subcase, the resource residual of the current solution is equal to $m\,|\Delta f|$. Thus, the algorithm allocates this amount to the non-upper set with the maximum positive increasing slope. For Δf sufficiently close to 0, the resource residual drops to 0 before the increasing slope of this set changes, and the algorithm terminates. If this is the unique lower set, it is clear that this move is equivalent to the application of Option C. If not, it is equivalent to the application of Option A. If the current solution does not have a non-upper set with positive increasing slope, the algorithm terminates and this is equivalent to the application of Option D. Note that the optimal solution of the modified LMCKE instance does not change when we add the constraints $C_k \geq L$ and $C_k \leq U'$. Additionally, the solution at hand satisfies all the constraints of the modified LMCKE instance, since the costs of all sets lie in an interval $[L, U']$, whose width does not exceed f'. Therefore, the current solution is optimal for the modified LMCKE instance, which implies that the minimum marginal decrease in P per unit decrease in f is achieved through the application of one of the options A, C or D. If case 2 is true, consider the associated LMCK instance, after adding the constraints $C_k \geq L'$ and $C_k \leq U$, for all $k \in S$. The order of the decision variables in the multiple choice lists and the master list is the same as in the original instance. The solution procedure allocates initially a resource amount equal to L' to each set. An extra resource amount equal to $n(L' - L)$ is required to increase the cost of the n lower sets from L to L'. We distinguish two subcases. The first subcase is when the resource residual at the optimal solution of the original LMCKE instance is equal to 0. Consider the non-lower set containing the variable that was increased last during the application of the algorithm on the original LMCKE instance. When the remaining resource is allocated to the sets in the current solution, the cost of this set assumes a value which is smaller than its original by $n(L' - L)$. For this solution, the resource residual is equal to 0, and the algorithm terminates. When the cost of a set is increased, its increasing slope at the beginning of this increase becomes its decreasing slope at the end. Since this set was increased last, it has the minimum decreasing slope among all non-lower sets in this solution. If this is the unique upper set, then clearly this is equivalent to the application of Option C. Otherwise, it is equivalent to the application of Option B. The second subcase that we consider is when the resource residual at the optimal solution of the original LMCKE instance is positive. The extra resource amount needed to increase the cost of the lower sets from L to L' is available in this subcase. Therefore, during the allocation of the remaining resource we get a solution in which the cost of each originally lower set is equal to L' and the cost of each originally non-lower set is the same as before. At that point, all lower sets have nonpositive increasing slopes, since they had nonpositive increasing slopes with smaller cost, too, and the increasing slopes of all internal sets remain nonpositive, since their cost is the same as before. Therefore, the algorithm terminates. This however is equivalent to the application of Option E. Note that the optimal solution of the modified LMCKE instance does not change when we add the constraints $C_k \geq L'$ and $C_k \leq U$. Additionally, the solution at hand satisfies all the constraints of the modified LMCKE instance, since the costs of all sets lie in an interval $[L', U]$, whose width does not exceed f'. Therefore, this solution is optimal for the modified LMCKE instance, which means that the minimum marginal decrease in P per unit decrease in f is achieved through the application of one of the options B, C or E. If case 3 is true, consider the associated LMCK instance, after adding the constraints $C_k \geq L'$ and $C_k \leq U'$, for all $k \in S$. We distinguish three subcases: 3a) $n(L' - L) = m(U - U')$, 3b) $n(L' - L) < m(U - U')$, and 3c) $n(L' - L) > m(U - U')$. Consider subcase 3a first. The order that the decision variables appear in the multiple choice lists and the master list is the same as before. During the application of the solution procedure, we get an intermediate solution in which the costs of the originally lower sets are equal to L', the costs of the originally internal sets are the same as before, and the costs of the originally upper sets are equal to U'. If the resource residual at the optimal solution of the original LMCKE instance is equal to 0, then this is true for this solution, too, since $n(L' - L) = m(U - U')$. If not, then all lower sets have nonpositive increasing slopes, since they had nonpositive increasing slopes with smaller costs, too, and the increasing slopes of all internal sets remain nonpositive, since their costs are the same as before. This implies that the current solution is optimal and the algorithm terminates, which means that the optimal solution in subcase 3a can be obtained through the application of Option C. Consider now subcase 3b. At some point during the application of the solution procedure we get an intermediate solution in which the costs of the lower sets are equal to L', the costs of the upper sets are equal to U' and the costs of the internal sets are the same as before. Since $n(L' - L) < m(U - U')$, the resource residual of this solution is positive. If the resource residual at the optimal solution of the original LMCKE instance is positive, then the algorithm terminates. This is because all lower sets have nonpositive increasing slopes, since they had nonpositive increasing slopes with smaller costs, too, and the increasing slopes of all internal sets remain nonpositive since their costs are the same as before. Let e be the sum of increasing slopes of the lower sets, a be the sum of decreasing slopes of the upper sets, $z = U - U'$ and $w = L' - L$. With respect to the optimal solution of the original LMCKE instance, ΔP is equal to $ew - az$, and Δf is equal to -$(w + z)$. Therefore, the "equivalent" ratio $\frac{\Delta P}{\Delta f}$ is equal to $\frac{az-ew}{w+z}$. Assume that the ratio of Option D, when applied to the optimal solution of the original LMCKE instance, is smaller than the ratio of Option C, i.e., that $a < \frac{an-em}{m+n} \iff am + an < an - em \Rightarrow a < -e$. Then, the ratio of Option D is also smaller than the "equivalent" ratio, since $a < \frac{az-ew}{w+z}$ becomes $aw + az < az - ew \iff a < -e$, which holds from above. If the ratio of Option D is not smaller than the ratio of Option C, we have $a \geq -e$. In this case, the "equivalent" ratio can not be smaller than the ratio of Option C, since from $\frac{an-em}{m+n} > \frac{az-ew}{w+z}$, we get $anw + anz - emw - emz > amz + anz - emw - enw \iff nw(a + e) > mz(a + e)$, which is a contradiction for subcase 3b, since $a + e \geq 0$. If the resource residual at the optimal solution of the original LMCKE instance is equal to 0, there may exist a non-upper set with positive increasing slope. Therefore, the positive resource residual, which is equal to $m(U-U')-n(L'-L) = mz-nw$, can be allocated to the non-upper set with the maximum increasing slope (let this slope be equal to o), in order to increase total profit. For Δf sufficiently close to 0, the budget residual drops to 0 before the increasing slope of this set changes and the algorithm terminates. With respect to the optimal solution of the original LMCKE instance, ΔP is equal to $ew - az + o(mz - nw)$, and Δf is equal to -$(w+z)$. Therefore, the "equivalent" ratio $\frac{\Delta P}{\Delta f}$ is equal to $\frac{(no-e)w+(a-mo)z}{w+z}$. Assume that the ratio of Option A, when applied to the optimal solution of the original LMCKE instance with the same increasing set, is smaller than the ratio of Option C, i.e. that $a-mo < \frac{an-em}{m+n} \iff am+an-m^2o-mno < an-em \iff a+e < o(m+n) \iff o > \frac{a+e}{m+n}$. Then, the ratio of Option A is also smaller than the "equivalent" ratio, since $a - mo < \frac{(no-e)w+(a-mo)z}{w+z}$ becomes $aw + az - mow - moz < now - ew + az - moz \iff a + e < o(m + n) \Rightarrow o > \frac{a+e}{m+n}$, which holds from above. If the ratio of Option A is not smaller than the ratio of Option C, we have $o \leq \frac{a+e}{m+n}$. Then, the "equivalent" ratio can not be smaller than the ratio of Option C, since $\frac{an-em}{m+n} > \frac{(no-e)w+(a-mo)z}{w+z}$ becomes $anw + anz - emw - emz > mnow + n^2ow - emw - enw + amz + anz - m^2oz - mnoz \iff anw + enw - mnow - n^2ow > emz + amz - m^2oz - mnoz \iff nw(a + e - mo - no) > mz(a + e - mo - no)$. Since $o \leq \frac{a+e}{m+n}$, the quantity in the parenthesis is nonnegative; therefore, this is a contradiction for subcase 3b. Note that the optimal solution of the modified LMCKE instance does not change when we add the constraints $C_k \geq L'$ and $C_k \leq U'$. Additionally, the solution at hand satisfies all the constraints of the modified LMCKE instance, since the costs of all sets lie in an interval $[L', U']$, whose width does not exceed f'. Therefore, the current solution is optimal for the modified LMCKE instance. Since we can also reach the optimal solution through the application of one of the options A, C or D, one of the five options always leads to the minimum marginal decrease in P per unit decrease in f in subcase 3b. For subcase 3c, consider the solution obtained when the cost of the originally lower sets becomes equal to L', the cost of each originally internal set equal to what it was at the optimal solution of the original LMCKE instance, and the cost of the originally upper sets equal to U'. Note that an extra resource amount equal to $n(L' - L)$ is needed to increase the cost of the lower sets from L to L' and an extra resource amount equal to $m(U - U')$ is recovered due to the fact that the cost of the upper sets is not increased beyond U'. If the resource residual at the optimal solution of the original LMCKE instance is positive, it can be used to cover the difference between $n(L' - L)$ and $m(U - U')$. At that point, the algorithm terminates and this is the optimal solution to the modified LMCKE instance. This is because all lower sets have nonpositive increasing slopes, since they had nonpositive increasing slope with smaller cost, too, and the increasing slopes of all internal sets remain nonpositive since their cost is the same as before. With respect to the optimal solution of the original LMCKE instance, ΔP is equal to $ew - az$, and Δf is equal to -$(w + z)$. Therefore, the "equivalent" ratio $\frac{\Delta P}{\Delta f}$ is equal to $\frac{az-ew}{w+z}$. Assume that the ratio of Option E, when applied to the optimal solution of the original LMCKE instance, is smaller than the ratio of Option C, i.e., that $-e < \frac{an-em}{m+n} \Rightarrow -em - en < an - em \Longleftrightarrow -e < a$. Then, the ratio of Option E is also smaller than the "equivalent" ratio, since $-e < \frac{az-ew}{w+z}$ becomes $-ew - ez < az - ew \Rightarrow -e < a$, which holds from above. If the ratio of Option E is not smaller than the ratio of Option C, we have $-e \geq a$. Then, the "equivalent" ratio can not be smaller than the ratio of Option C, since $\frac{an-em}{m+n} > \frac{az-ew}{w+z}$ becomes $anw + anz - emw - emz > amz + anz - emw - enw \Longleftrightarrow nw(a+e) > mz(a+e)$, which is a contradiction for subcase 3c, since $a + e \leq 0$. If the resource residual at the optimal solution of the original LMCKE instance is equal to 0, then this solution is superoptimal for the modified LMCKE instance, because it requires an extra resource amount equal to $n(L' - L) - m(U - U')$. Therefore, the optimal solution can be obtained by decreasing the cost of the non-lower set with the minimum decreasing slope by this amount. If this slope is equal to u, ΔP is equal to $ew - az - u(nw - mz)$, and Δf is equal to -$(w + z)$. Therefore, the "equivalent" ratio $\frac{\Delta P}{\Delta f}$ is equal to $\frac{(nu-e)w+(a-mu)z}{w+z}$. Assume that the ratio of Option B, when applied to the optimal solution of the original LMCKE instance with the same decreasing set, is smaller than the ratio of Option C, i.e. that $nu - e < \frac{an-em}{m+n} \Rightarrow mnu - em + n^2u - en < an - em \Longleftrightarrow a + e > u(m + n) \Rightarrow u < \frac{a+e}{m+n}$. Then, the ratio of Option B is also smaller than the "equivalent" ratio, since $nu - e < \frac{(nu-e)w+(a-mu)z}{w+z}$ becomes $nuw - ew + nuz - ez < nuw - ew + az - muz \Longleftrightarrow a + e > u(m + n) \Rightarrow u < \frac{a+e}{m+n}$, which holds from above. If the ratio of Option B is not smaller than the ratio of Option C, we have $u \geq \frac{a+e}{m+n}$. In this case, the "equivalent" ratio can not be smaller than the ratio of Option C, since $\frac{an-em}{m+n} > \frac{(nu-e)w+(a-mu)z}{w+z}$ becomes $anw + anz - emw - emz > mnuw + n^2uw - emw - enw + amz + anz - m^2uz - mnuz \Longleftrightarrow anw - mnuw - n^2uw + enw > emz + amz - m^2uz - mnuz \Longleftrightarrow nw(a + e - mu - nu) > mz(a + e - mu - nu)$. Since $u \geq \frac{a+e}{m+n}$, the quantity in the parenthesis is nonpositive; therefore, this is a contradiction for subcase 3c. Note that the optimal solution of the modified LMCKE instance does not change when we add the constraints $C_k \geq L'$ and $C_k \leq U'$. Additionally, the solution at hand satisfies all the constraints of the modified LMCKE instance, since the costs of all sets lie in an interval $[L', U']$, whose width does not exceed f'. Therefore, the current solution is optimal for the modified LMCKE instance. Since we can also reach the optimal solution through the application of one of the options B, C or E, one of the five options always leads to the minimum marginal decrease in P per unit decrease in f in subcase 3c. $\square$